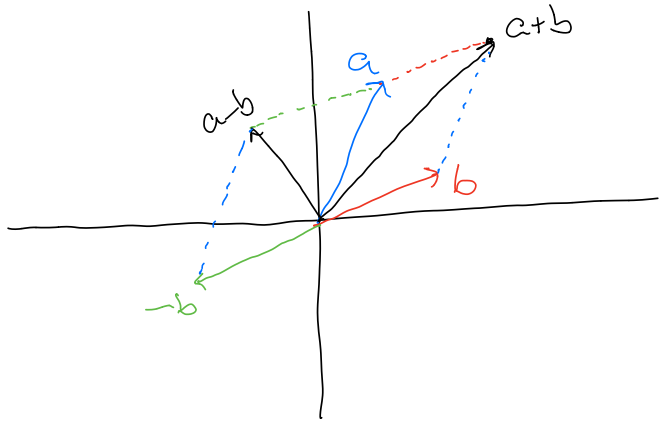
Geometric view of vector addition and subtraction.

Suppose a and b are d-dimensional vectors, and I want to calculate $a+b$ and $a-b$.

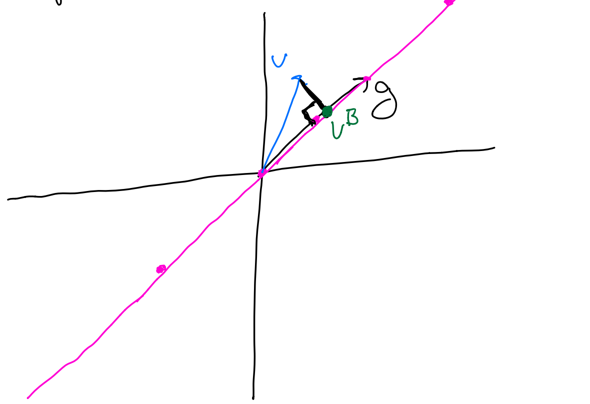
"Parallelogram Rule" Example with $d=2$



Orthogonal Projections

Suppose B is a linear subspace of $\mathbb{R}^d$ spanned by the vector g.

Example with $d=2$:



$B = \{v : v = c \cdot g$ for a constant $c \in \mathbb{R}\}$

The orthogonal projection of a vector v onto the subspace B is calculated as

$$v^B = \boxed{\frac{g g^T}{g^T g}} \cdot v$$

↑ projection matrix P

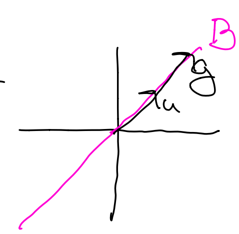
Justification of this formula.

First, note that $g^T g = g \cdot g = g_1^2 + g_2^2 + \cdots + g_d^2$

$$= \left(\sqrt{g_1^2 + g_2^2 + \cdots + g_d^2}\right)^2$$

$$= (\|g\|)^2$$

So $P = \dfrac{g g^T}{g^T g} = \dfrac{g g^T}{(\|g\|)^2} = \left(\dfrac{g}{\|g\|}\right)\left(\dfrac{g}{\|g\|}\right)^T = u u^T$

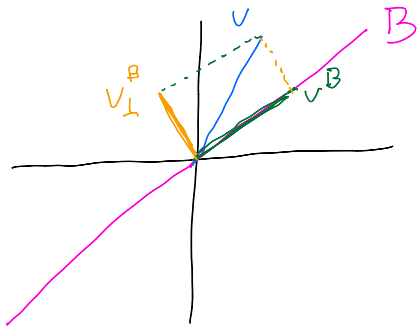


where u is a unit vector

Write $v = v^B + v_\perp^B$

where v^B is the projection of v into B

and $v_\perp^B$ is the component of v that is orthogonal to B.



Goal is to show

$$v^B = \frac{gg^T}{g^Tg} v = uu^T(v^B + v_\perp^B) = uu^Tv^B + \cancel{uu^Tv_\perp^B}$$

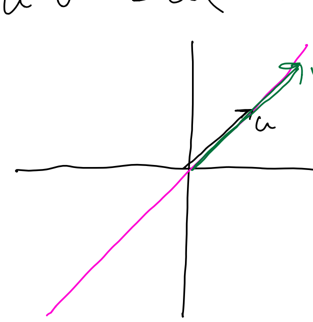
We will be done if

$$\checkmark\ \underline{uu^Tv^B = v^B} \quad \text{and} \quad \boxed{uu^Tv_\perp^B = 0}$$

To show this, use:

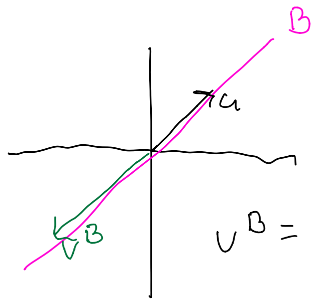
$$x \cdot y = \|x\| \cdot \|y\| \cdot \cos(\theta)$$

$$uu^Tv^B = u(u \cdot v^B) = \underline{u\left(\overset{1}{\cancel{\|u\|}} \cdot \|v^B\| \cdot \cos(\theta)\right)}$$



$\Rightarrow v^B = u \cdot \|v^B\| \cdot 1$ (since $\frac{v_B}{\|v_B\|} = u$)

$\cos(\theta)$ if u and v^B point in same direction



$v^B = u \cdot \|v^B\| \cdot -1$

cos(θ) if u and v^B point in opposite directions.

Last step: show $uu^T v_\perp^B = 0$

$$uu^T v_\perp^B = u\cdot(\underbrace{u\circ v_\perp^B}_{0}) = u\cdot 0 = 0.$$

0 since u and $v_\perp^B$ are orthogonal!