Stat 343: Quiz 1

What's Your Name? Solutions

Note: partial credit will be awarded. If you don't know how to do the full problem, write down the set up.

Two different companies are planning to conduct surveys to estimate the proportion θ of likely voters in the Massachusetts Democratic primaries who prefer Elizabeth Warren. The surveys will have different sample sizes, and so the resulting estimators will have different variances. Both will use carefully designed sampling strategies, so it is reasonable to think that both surveys will give unbiased estimates of θ. Additionally, both sample sizes will be large enough that it would be reasonable to model the estimators as being normally distributed. To sum up, we have:

$X_1 \sim \text{Normal}(\theta, \sigma_1^2)$ **and**

$X_2 \sim \text{Normal}(\theta, \sigma_2^2)$,

where X_1 is the estimate of θ from the first survey and X_2 is the estimate of θ from the second survey. Note that the two distributions have the same mean, but different variances. For the purpose of this question, assume the estimates from the two surveys are independent and the variances σ_1^2 and σ_2^2 are known constants.

The pdf of a normally distributed random variable with mean θ is:

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-1}{2\sigma^2}(x-\theta)^2\right]$

1. Based on the two poll results, find the maximum likelihood estimator of θ. Your likelihood function should be a function of θ, based on the poll results x_1 and x_2. Be sure to check that your estimator maximizes the likelihood function.

Likelihood function:

$$\mathcal{L}(\theta | x_1, x_2) = f_{X_1,X_2}(x_1, x_2 | \theta)$$
$$= f_{X_1}(x_1|\theta) \cdot f_{X_2}(x_2|\theta)$$
$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left[\frac{-1}{2\sigma_1^2}(x_1-\theta)^2\right] \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} \exp\left[\frac{-1}{2\sigma_2^2}(x_2-\theta)^2\right]$$
$$= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[\frac{-1}{2\sigma_1^2}(x_1-\theta)^2 - \frac{1}{2\sigma_2^2}(x_2-\theta)^2\right]$$

Log-likelihood:

$$\ell(\theta | x_1, x_2) = \log\left[\frac{1}{2\pi\sigma_1\sigma_2}\right] - \frac{1}{2\sigma_1^2}(x_1-\theta)^2 - \frac{1}{2\sigma_2^2}(x_2-\theta)^2$$

Set derivative of log-likelihood equal to 0:

$$0 = \frac{d}{d\theta} \ell(\theta | x_1, x_2) = \cancel{+}\frac{1}{\cancel{2}\sigma_1^2} \cdot \cancel{2}(x_1 - \theta)\cancel{(-1)} + \frac{1}{\cancel{2}\sigma_2^2} \cdot \cancel{2}(x_2 - \theta)\cancel{(-1)}$$

$$\Rightarrow \frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2} = \theta\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)$$

$$\left.\begin{aligned} \Rightarrow \theta &= \frac{\frac{1}{\sigma_1^2} x_1 + \frac{1}{\sigma_2^2} x_2}{(1/\sigma_1^2 + \frac{1}{\sigma_2^2})} \\ &= \frac{\sigma_2^2 x_1 + \sigma_1^2 x_2}{\sigma_1^2 + \sigma_2^2} \end{aligned}\right\} \text{a critical point of the log-likelihood function}$$

Check whether we have a maximum:

$$\frac{d^2}{d\theta^2} \ell(\theta | x_1, x_2) = \underbrace{\frac{-1}{\sigma_1^2} - \frac{1}{\sigma_2^2}}_{\text{negative since } \sigma_1^2 > 0 \text{ and } \sigma_2^2 > 0.} < 0$$

The critical point we found above is a maximum of the log-likelihood function.

State answer as a random variable

The maximum likelihood estimator of θ is

$$\hat{\theta}^{MLE} = \frac{\frac{1}{\sigma_1^2} X_1 + \frac{1}{\sigma_2^2} X_2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{\sigma_2^2 X_1 + \sigma_1^2 X_2}{\sigma_1^2 + \sigma_2^2}$$

(Use capital X's, either expression is fine.