Significance level of a test: α

Reject H_0 if p-value $< \alpha$

Size: $P(\text{Type I Error} \mid \underline{H_0 \text{ true}}) \le \alpha$

$= P(\text{Reject } H_0 \mid H_0 \text{ true})$

$= \int_R \cdots \int f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \underline{\theta_0})\, d\underset{\sim}{x} = \int_R \cdots \int f_{X_1,\ldots,X_n}(x_1,\ldots,x_n \mid \theta_0)\, dx_1 \cdots dx_n$

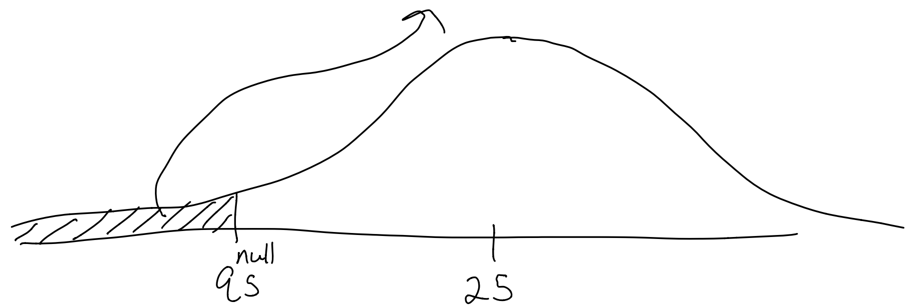
R is the rejection region: the set of values $x_1, \ldots, x_n$ for which we would reject the test.

$$\boxed{= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{I}_R(\underset{\sim}{x}) \cdot f_{\underset{\sim}{X}}(\underset{\sim}{x} \mid \theta_0)\, d\underset{\sim}{x}}$$

1 if $\underset{\sim}{x} \in R$, 0 if $x \notin R$.

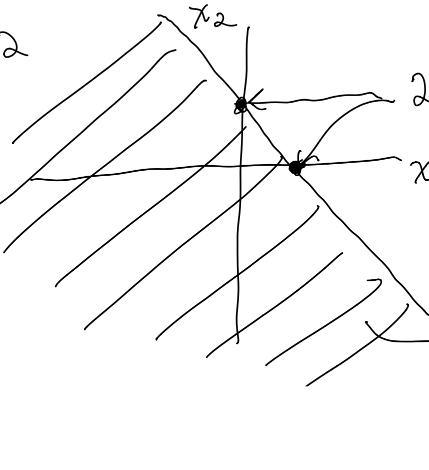
Example:

$P(\text{Type I Error} \mid H_0 \text{ true}) = \int_{-\infty}^{q_5^{null}} f_{\bar{X}|\theta}(\bar{x} \mid 25)\, d\bar{x}$



$R = \{(x_1, \ldots, x_n) : \bar{x} \le q_5^{null}\}$

If $n = 2$



$2q_5^{null}$, so $\frac{x_1 + x_2}{2} = q_5^{null}$

shaded region is R

β: P(Type II Error | H_0 false) = P(fail to reject H_0 | H_0 false)

Power $= 1-\beta =$ P(Reject H_0 | $\underline{H_0 \text{ false}}$)

$$= \int \cdots \int_R f_{\underset{\sim}{X}|\Theta}(\underset{\sim}{x} \mid \underline{\Theta_A})\, d\underset{\sim}{x}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbb{I}_R(\underset{\sim}{x}) \cdot f_{\underset{\sim}{X}|\Theta}(\underset{\sim}{x} \mid \Theta_A)\, d\underset{\sim}{x}$$

Neyman-Pearson Lemma

Suppose H_0 and H_A are both simple hypothesis ($H_0: \Theta = \Theta_0$, $H_A: \Theta = \Theta_A$) and that the test that rejects H_0 whenever the likelihood ratio statistic is less than ω^* has size $\underline{\alpha}$.

Then any other test with size $\leq \alpha$ has power $\leq$ power of the L.R.T.

Proof: Denote the rejection set for LRT by R^{LRT}, and for the other test by R^{other}

$$\mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) = \begin{cases} 1 & \text{if we reject } H_0 \text{ based on LRT: } \frac{f(\underset{\sim}{x} \mid \Theta_0)}{f(\underset{\sim}{x} \mid \Theta_A)} < \omega^*, \text{ or } \omega^* \cdot f(\underset{\sim}{x} \mid \Theta_A) - f(\underset{\sim}{x} \mid \Theta_0) > 0 \\ 0 & \text{if we fail to reject } H_0. \end{cases}$$

$$\mathbb{I}_{R^{other}}(\underset{\sim}{x}) = \begin{cases} 1 & \text{if other test rejects } H_0 \\ 0 & \text{if not.} \end{cases}$$

Goal: Show Power other test $\le$ Power LRT

$$\iff \underbrace{\int\cdots\int \mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \cdot f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_A)\, d\underset{\sim}{x}}_{\text{power of other tes}} \le \int\cdots\int \mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) \cdot f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_A)\, d\underset{\sim}{x}$$

$$\iff 0 \le \int\cdots\int \mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) \cdot f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_A)\, d\underset{\sim}{x} - \int\cdots\int \mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) f_{\underset{\sim}{X}}(\underset{\sim}{x}|\theta_A)\, d\underset{\sim}{x}$$

Step 1: Note

$$\mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \cdot \{w^* \cdot f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)\} \le \mathbb{I}_{R^{LRT}} \cdot \{w^* \cdot f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)\} \quad \star$$

Check in 2 cases:

Case 1: $\mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) = 1$. This means $\underline{w^* f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)} > 0$

Divide both sides of ($\star$) by $w^* f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)$

$\mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \le 1$ ✓ which is true.

Case 2: $\mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) = 0$. This means $w^* f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0) \le 0$

Continue from $\star$:

$\mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \cdot (\# \le 0) \le 0 \cdot (\# \le 0)$

Step 2: Integrate both sides of $\star$:

$$\int\cdots\int \mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \cdot \{w^* f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)\} d\underset{\sim}{x}$$

$$\le \int\cdots\int \mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) \{w^* f(\underset{\sim}{x}|\theta_A) - f(\underset{\sim}{x}|\theta_0)\} d\underset{\sim}{x}$$

$$\iff \underbrace{w^* \int\cdots\int \mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) f(\underset{\sim}{x}|\theta_A) d\underset{\sim}{x}}_{w^* \cdot \text{Power other test}} - \underbrace{\int\cdots\int \mathbb{I}_{R^{\text{other}}}(\underset{\sim}{x}) \cdot f(\underset{\sim}{x}|\theta_0) d\underset{\sim}{x}}_{\text{size of other test} \le \alpha}$$

$$\le \underbrace{w^* \int\cdots\int \mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) \cdot f(\underset{\sim}{x}|\theta_A)\, d\underset{\sim}{x}}_{w^* \cdot \text{power of LRT}} - \underbrace{\int\cdots\int \mathbb{I}_{R^{LRT}}(\underset{\sim}{x}) f(\underset{\sim}{x}|\theta_0) d\underset{\sim}{x}}_{\text{size of LRT} = \alpha}$$

$$\iff \underbrace{(\text{size of LRT})}_{\alpha} - \underbrace{(\text{size of other test})}_{\leq \alpha} \leq w^* \cdot (\text{power of LRT}) - w^* \cdot (\text{power other test})$$

$$\underbrace{(\text{size of LRT}) - (\text{size of other test})}_{\text{must be } \geq 0}$$

$$\Rightarrow \quad 0 \leq \cancel{w^*}\left\{(\text{power of LRT}) - (\text{power other test})\right\}$$

w^* is a critical value for likelihood ratio

$$\Rightarrow \quad \text{power of LRT} \geq \text{power of other test.}$$