Warm Up: Power Functions for Hypothesis Tests

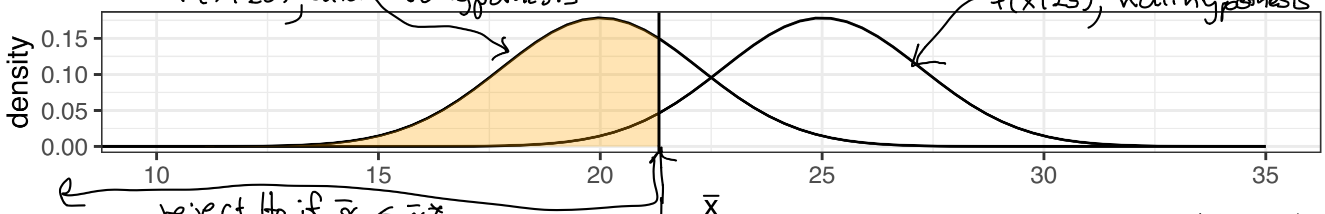
- Data Model: $X_1, \dots, X_5 \overset{\text{i.i.d.}}{\sim} \text{Normal}(\theta, 5^2)$
- We saw that the likelihood ratio test is *equivalent* to a test based on $\bar{x}$. The p-value is $P(\bar{X} \le \bar{x} | \theta = 25)$ ("extreme" values of $\bar{x}$ are those that are at least as small as $\bar{x}$)
- The *power* of the test is $P(\text{reject } H_0 | H_0 \text{ incorrect})$ $= P(\bar{X} \le \bar{x}^* \mid H_0 \text{ incorrect})$
  $= P(\bar{X} \le \bar{x}^* \mid \theta = 20)$

1. Consider a test of the hypotheses $H_0 : \theta = 25$ vs. $H_A : \theta = 20$. Below is a picture showing the pdf $f_{\bar{X}|\theta}(\bar{x}|20)$ of a Normal$(20, 5^2/5)$ distribution and the pdf $f_{\bar{X}|\theta}(\bar{x}|25)$ of a Normal$(25, 5^2/5)$ distribution, along with a vertical line at q_5^{null}, the 5th percentile of the Normal$(25, 5^2/5)$ distribution.

- **Shade in the area corresponding to $1 - \beta$, the power of the likelihood ratio test if H_A is correct.**
- **Show how you would calculate the power of the test as an integral of either $f_{\bar{X}|\theta}(\bar{x}|20)$ or $f_{\bar{X}|\theta}(\bar{x}|25)$**

$f(\bar{x}|20)$, alternative hypothesis

$f(\bar{x}|25)$, null hypothesis



reject H_0 if $\bar{x} \le \bar{x}^*$

$\bar{x}^* =$ 5th percentile of dist'n of $\bar{X}$ if H_0 correct

$1 - \beta = \int_{-\infty}^{q_5^{null}} f_{\bar{X}|\theta}(\bar{x}|20) d\bar{x}$

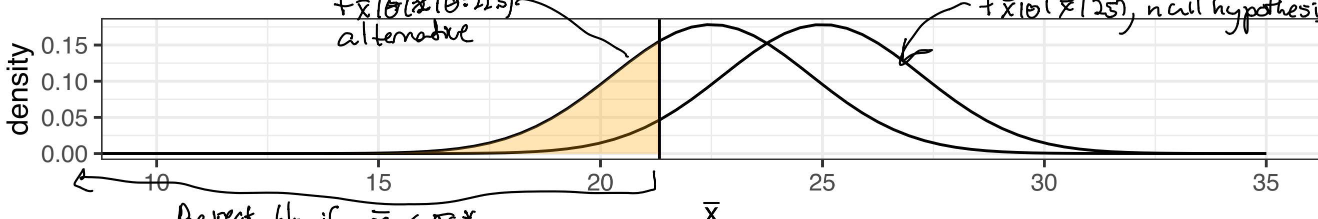
$$\text{Power} = P(\bar{X} \le \bar{x}^* | \theta = 20) = \int_{-\infty}^{\bar{x}^*} f_{\bar{X}|\theta}(\bar{x}|20)\, d\bar{x}$$

2. Suppose that instead we were testing the hypotheses $H_0 : \theta = 25$ vs. $H_A : \theta = 22.5$. Below is a picture showing the pdf $f_{\bar{X}|\theta}(\bar{x}|22.5)$ of a Normal$(22.5, 5^2/5)$ distribution and the pdf $f_{\bar{X}|\theta}(\bar{x}|25)$ of a Normal$(25, 5^2/5)$ distribution, along with a vertical line at q_5^{null}, the 5th percentile of the Normal$(25, 5^2/5)$ distribution.

- **Shade in the area corresponding to $1 - \beta$, the power of the likelihood ratio test if H_A is correct.**
- **Show how you would calculate the power of the test as an integral of either $f_{\bar{X}|\theta}(\bar{x}|22.5)$ or $f_{\bar{X}|\theta}(\bar{x}|25)$**

$f_{\bar{X}|\theta}(\bar{x}|\theta = 22.5)$
alternative

$f_{\bar{X}|\theta}(\bar{x}|25)$, null hypothesis



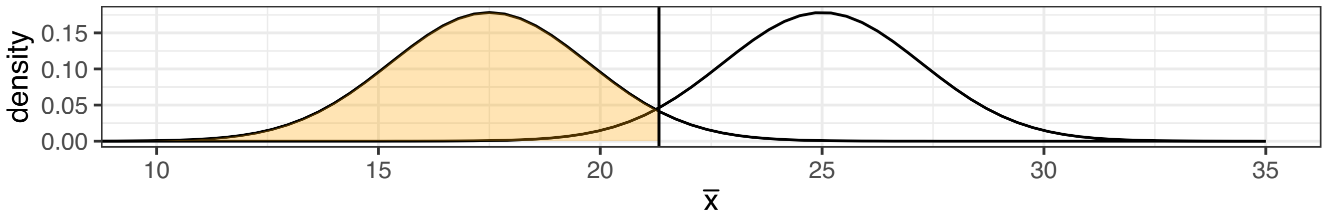
Reject H_0 if $\bar{x} \le \bar{x}^*$

$1 - \beta = \int_{-\infty}^{q_5^{null}} f_{\bar{X}|\theta}(\bar{x}|22.5) d\bar{x}$

Note: power of test is smaller if θ from alternative hypothesis is closer to θ from null hypothesis.

3. Suppose that instead we were testing the hypotheses $H_0 : \theta = 25$ vs. $H_A : \theta = 17.5$. Below is a picture showing the pdf $f_{\bar{X}|\theta}(\bar{x}|17.5)$ of a Normal$(17.5, 5^2/5)$ distribution and the pdf $f_{\bar{X}|\theta}(\bar{x}|25)$ of a Normal$(25, 5^2/5)$ distribution, along with a vertical line at q_5^{null}, the 5th percentile of the Normal$(25, 5^2/5)$ distribution.

- **Shade in the area corresponding to $1 - \beta$, the power of the likelihood ratio test if H_A is correct.**
- **Show how you would calculate the power of the test as an integral of either $f_{\bar{X}|\theta}(\bar{x}|17.5)$ or $f_{\bar{X}|\theta}(\bar{x}|25)$**



$1 - \beta = \int_{-\infty}^{q_5^{null}} f_{\bar{X}|\theta}(\bar{x}|17.5) d\bar{x}$

4. For which of the alternative hypotheses above ($\theta = 17.5$, $\theta = 20$, or $\theta = 22.5$) is the power of the test largest? For which is the power smallest?

The power is largest for $\theta = 17.5$ and smallest for $\theta = 22.5$.

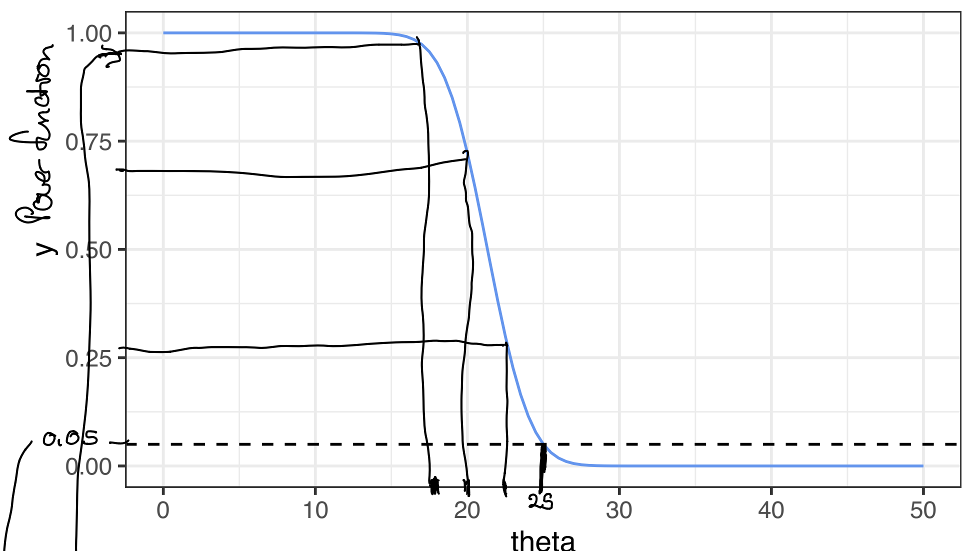
Observation: The power of the test depends on the alternative hypothesis

Definition: power function In previous examples, $R = (-\infty, q_5^{null}]$

The *power function* $K(\theta)$ for a test is the power of the test at θ:

$K(\theta) = \int_R f_{\mathbf{X}|\theta}(\mathbf{x}|\theta) d\mathbf{x} \approx \int_{-\infty}^{q_5^{null}} f_{\bar{X}|\theta}(\bar{x}|\theta) d\bar{x}$

where R denotes the rejection region of the test (i.e., R is the set of $\mathbf{x}$ such that the p-value is less than α)



If $\theta = 17.5$, very high power $\rightarrow$ high prob. reject H_0.

null hypothesis

If $\theta = 25$, $P(\text{reject } H_0 | \theta = 25) = 0.05$

Type I Error rate, size of test