

Hypothesis Testing - Normal Example

We have 2 batches of paint, one of which is quick-dry; quick-dry paint will dry in an average of 10 minutes, and regular paint in an average of 25 minutes. The paint is unlabeled, and we forgot which was which! We paint 5 boards from batch 1 and record the drying time. We believe that the drying times are normally distributed with a standard deviation of 5 minutes for both paint types.

We have a vague feeling that batch 1 is quick-dry; we will calculate a p-value to see how strong the evidence is that the batch we used is *not* 25 minutes:

$$H_0 : \theta = 25$$

$$H_A : \theta = 10$$

(a) Find the form of the likelihood ratio statistic for this test.

You will need to use the fact that if $X_i \sim \text{Normal}(\theta, \sigma^2)$ then its pdf is $f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(x_i - \theta)^2\right]$. In this case, we are assuming $\sigma = 5$; the value of θ will be different in the numerator and denominator of the likelihood ratio statistic.

(b) Show that the likelihood ratio statistic can be simplified to a form that involves the sample mean \bar{X} .

(c) Show how the p-value for the likelihood ratio test could be calculated based on an observed sample mean of \bar{x} . Your answer should be a probability involving the random variable \bar{X} and the observed value \bar{x} based on the observed sample data.

(d) Show that your p-value calculation from part (c) is equivalent to $P(\bar{X} \leq \bar{x})$

(e) Is a small or large value of \bar{x} evidence against H_0 ? How is this determined by the set up of the null and alternative hypotheses?