Confidence Interval for Variance of a normal distribution

Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Normal}(\mu, \sigma^2)$ with both μ and σ^2 unknown.

(a) We have previously stated that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$. Explain in a sentence or two why this is a pivotal quantity for σ^2.

The quantity $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sigma^2}$ is a function of the observable data $X_1, \ldots, X_n$ and σ^2, and the only parameter of its distribution is the degrees of freedom $n-1$. We know the sample size, so the distribution of $\frac{(n-1)S^2}{\sigma^2}$ does not depend on any unknown parameters.

(b) Use the pivotal quantity from part (a) to find a 2-sided confidence interval for σ^2. At this point we are not working with a sample of observed data so your interval endpoints should be random variables.

Hint: Note that if $\frac{1}{5} < \frac{1}{4} < \frac{1}{3}$, you can take the reciprocal of all three terms if you also reverse the direction of the inequalities: $5 > 4 > 3$.

$$\begin{aligned}
1 - \alpha &= P\left(\chi^2_{n-1}(\frac{\alpha}{2}) \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi^2_{n-1}(1 - \frac{\alpha}{2})\right) \\
&= P\left(\frac{1}{\chi^2_{n-1}(\frac{\alpha}{2})} \geq \frac{\sigma^2}{(n-1)S^2} \geq \frac{1}{\chi^2_{n-1}(1 - \frac{\alpha}{2})}\right) \\
&= P\left(\frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})} \geq \sigma^2 \geq \frac{(n-1)S^2}{\chi^2_{n-1}(1 - \frac{\alpha}{2})}\right) \\
&= P\left(\frac{(n-1)S^2}{\chi^2_{n-1}(1 - \frac{\alpha}{2})} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})}\right)
\end{aligned}$$

Therefore, a $(1-\alpha) * 100\%$ confidence interval for σ^2 is $\left[\frac{(n-1)S^2}{\chi^2_{n-1}(1-\frac{\alpha}{2})}, \frac{(n-1)S^2}{\chi^2_{n-1}(\frac{\alpha}{2})}\right]$.

Unlike most confidence intervals you've seen in other classes, this interval is not of the form

(Estimate) $\pm$ (Margin of Error).