Thm: As $n \to \infty$, the posterior distribution converges in probability to a Normal$\left(\hat{\theta}^{MLE}, \frac{1}{J(\hat{\theta}^{MLE})}\right)$

- Reminder 1: $J(\theta^*) = -\frac{d^2}{d\theta^2} \ell(\theta \mid x_1, \dots, x_n)\Big|_{\theta=\theta^*}$

$$= -\ell''(\theta^* \mid x_1, \dots, x_n)$$

- Reminder 2: If $Y \sim$ Normal(μ, σ^2) then its pdf is

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(y-\mu)^2\right)$$

Our goal is to show the posterior pdf is:

$$f_{\Theta \mid X_1, \dots, X_n}(\theta \mid x_1, \dots, x_n) = \frac{1}{\sqrt{2\pi \frac{1}{J(\hat{\theta}^{MLE})}}} \cdot \exp\left(\frac{-1}{2\frac{1}{J(\hat{\theta}^{MLE})}} (\theta - \hat{\theta}^{MLE})^2\right)$$

- Reminder 3: If n is large, $\log\{f_\Theta(\theta)\}$ is approximately constant (relative to log-likelihood)

$$\frac{d}{d\theta} \log\{f_\Theta(\theta)\} = 0 \quad \text{and} \quad \frac{d^2}{d\theta^2} \log\{f_\Theta(\theta)\} = 0$$

Main idea of proof: Take a 2nd order Taylor approximation to the log of the posterior distribution.

$$\rightarrow f_{\Theta|X_1,\ldots,X_n}(\theta|x_1,\ldots,x_n) = \underbrace{c \cdot f_{\Theta}(\theta) \cdot f_{X_1,\ldots,X_n|\Theta}(x_1,\ldots,x_n)}$$

$$= \exp\left[\underbrace{\log\{c \cdot f_{\Theta}(\theta) \cdot f_{X_1,\ldots,X_n|\Theta}(x_1,\ldots,x_n|\theta)\}}\right]$$

$$= \exp\left[\log(c) + \log\{f_{\Theta}(\theta)\} + \underbrace{\log\{f_{X_1,\ldots,X_n|\Theta}(x_1,\ldots,x_n|\theta)\}}_{\ell(\theta|x_1,\ldots,x_n)}\right] \quad (1)$$

2nd order Taylor approx. to insides of $\exp(\cdots)$ at $\hat{\theta}^{MLE}$

constant term of Taylor approx.

linear term

$$P_2(\theta) = \boxed{\log(c) + \log\{f_{\Theta}(\hat{\theta}^{MLE})\} + \ell(\hat{\theta}^{MLE}|x_1,\ldots,x_n)}$$

$$\boxed{+ \frac{d}{d\theta}\left[\cancel{\log(c)} + \cancel{\log\{f_{\Theta}(\theta)\}} + \ell(\theta|x_1,\ldots,x_n)\right]\Big|_{\theta=\hat{\theta}^{MLE}}(\theta-\hat{\theta}^{MLE})}$$

$$+ \frac{1}{2}\frac{d^2}{d\theta^2}\left[\cancel{\log(c)} + \cancel{\log\{f_{\Theta}(\theta)\}} + \ell(\theta|x_1,\ldots,x_n)\right]\Big|_{\theta=\hat{\theta}^{MLE}}(\theta-\hat{\theta}^{MLE})^2$$

$$= c_1 + \underline{\cancel{\ell'(\hat{\theta}^{MLE}|x_1,\ldots,x_n)}}\,(\theta-\hat{\theta}^{MLE})$$

$$+ \frac{1}{2}\underbrace{\ell''(\hat{\theta}^{MLE}|x_1,\ldots,x_n)}(\theta-\hat{\theta}^{MLE})^2$$

$$= c_1 - \frac{1}{2}\{\underbrace{-\ell''(\hat{\theta}^{MLE}|x_1,\ldots,x_n)}\}(\theta-\hat{\theta}^{MLE})^2$$

$$= c_1 - \frac{1}{2}J(\hat{\theta}^{MLE})(\theta-\hat{\theta}^{MLE})^2 \quad (2)$$

Plug (2) into (1):

$$f_{\Theta|X_1,\ldots,X_n}(\theta|x_1,\ldots,x_n) \approx \exp\left[c_1 - \frac{1}{2}J(\hat{\theta}^{MLE})(\theta-\hat{\theta}^{MLE})^2\right]$$

$$= \underline{\exp(c_1)} \cdot \exp\left(\frac{-1}{2\,\underline{\frac{1}{J(\hat{\theta}^{MLE})}}}(\theta-\underline{\hat{\theta}^{MLE}})^2\right)$$

End of proof! :)

Laplace Approximation used more commonly:

$$\Theta \mid \underline{X}_1, \ldots, \underline{X}_n \sim \text{Normal}\left(\theta^{mode}, \frac{1}{J(\theta^{mode})}\right)$$