Section 8.6 Example B

• Suppose $X_1, \dots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$

$$f_{X_i}(x_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

• For mathematical and notational convenience, let's set

$\theta = \mu$ and $\xi = \frac{1}{\sigma^2}$. ξ is the precision: a high precision means a small variance.

• $f_{X_i}(x_i|\theta, \xi) = \left(\frac{\xi}{2\pi}\right)^{1/2} \exp\left(-\frac{1}{2}\xi(x-\theta)^2\right)$

Case 1: Unknown mean, known variance

Conjugate prior for Θ:

$$\Theta \sim \text{Normal}(\overset{\gamma_0}{\cancel{\theta_0}}, \xi_0^{-1})$$

$$f_{\Theta|X_1,\dots,X_n}(\theta|x_1,\dots,x_n) = c \cdot f_\Theta(\theta) \cdot f_{X_1,\dots,X_n|\Theta}(x_1,\dots,x_n|\theta)$$

$$= c \cdot \left(\frac{\xi_0}{2\pi}\right)^{1/2} \exp\left\{\frac{-1}{2}\xi_0(\theta - \overset{\gamma_0}{\cancel{\theta_0}})^2\right\} \cdot \prod_{i=1}^n \left(\frac{\xi}{2\pi}\right)^{1/2} \exp\left\{\frac{-1}{2}\xi(x_i-\theta)^2\right\}$$

$$= c \cdot \exp\left\{\frac{-1}{2}\left(\xi_0(\theta^2 - 2\theta\gamma_0 + \gamma_0^2)\right) + \frac{-1}{2}\xi \cdot \sum_{i=1}^n (x_i-\theta)^2\right\}$$

$$= c \cdot \exp\left\{\frac{-1}{2}\left(\xi_0\theta^2 - 2\xi_0\theta\gamma_0\right) + \frac{-1}{2}\xi \cdot \sum(x_i-\bar{x})^2 + \frac{-1}{2}\xi \cdot n(\bar{x}-\theta)^2\right\}$$

$$= c \cdot \exp\left\{\frac{-1}{2}\left(\xi_0\theta^2 - 2\xi_0\theta\gamma_0 + n\theta^2 - 2n\bar{x}\theta\right)\right\}$$

$$\Big\downarrow b$$

$$\propto \exp\left\{\frac{-1}{2}\xi_{post}(\theta - \gamma_{post})^2\right\}$$

where $\xi_{post} = n\xi + \xi_{prior}$

$$\gamma_{post} = \frac{n\xi\bar{x} + \gamma_{prior}\xi_{prior}}{n\xi + \xi_{prior}} = \bar{x}\frac{n\xi_0}{n\xi_0 + \xi_{prior}} + \theta_0 \cdot \frac{\xi_{prior}}{n\xi_0 + \xi_{prior}}$$

Take cases when $\xi_{prior} \to 0$, $n \to \infty$

$$\sum(x_i-\theta)^2 = \sum(x_i-\bar{x}+\bar{x}-\theta)^2$$

$$= \sum(x_i-\bar{x})^2 + 2\sum(x_i-\bar{x})(\bar{x}-\theta) + \sum(\bar{x}-\theta)^2$$

$$= \sum(x_i-\bar{x})^2 + \sum(\bar{x}-\theta)^2$$

Normal, known mean & unknown variance/precision

Prior for Ξ: Gamma(α, λ)

$$f_{\Xi|X}(\xi|x) = c \cdot f_{\Xi}(\xi) \cdot f_{X|\Xi}(x|\xi)$$

$$= c \cdot \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \xi^{\alpha-1} e^{-\lambda\xi} \cdot \prod_{i=1}^{n} \left(\frac{\xi}{2\pi}\right)^{1/2} \exp\left\{\frac{-\xi}{2}(x_i-\theta)^2\right\}$$

$$\propto \xi^{\alpha+1} \xi^{n/2} e^{-\lambda\xi} e^{-\frac{\xi}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$$

$$= \xi^{\alpha+\frac{n}{2}-1} e^{-\xi\left(\lambda+\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)}$$

$$\Xi|X_1,\dots,X_n \sim \text{Gamma}\left(\alpha+\frac{n}{2},\ \lambda+\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2\right)$$

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unknown mean and variance:

$\Theta \sim$ Normal(γ_0, ξ_0^{-1})

$\Xi \sim$ Gamma(α, λ)

Θ, Ξ independent in prior.

$$f_{\Theta,\Xi|X_1,\dots,X_n}(\theta,\xi|x_1,\dots,x_n) \propto f_{\Theta}(\theta)\, f_{\Xi}(\xi) \cdot f_{X|\Theta,\Xi}(x_1,\dots,x_n|\theta,\xi)$$

$$\propto \xi_0^{1/2} \exp\left\{\frac{-1}{2}\xi_0(\theta-\gamma_0)^2\right\}$$
$$\times\ \xi^{\alpha-1} e^{-\lambda\xi}$$
$$\times\ \prod_{i=1}^{n}\left(\frac{\xi}{2\pi}\right)^{1/2} \exp\left\{\frac{-\xi}{2}(x_i-\theta)^2\right\}$$

Cannot be recognized as a known distribution

Integral to find constant of pdf