Recall our first maximum likelihood example:

- 16 babies showed a video of a character trying to go up a hill.
- Each baby chose to play with either the helpful toy or the unhelpful toy.
- 14 babies chose the helpful toy

- Our model was

$$X \sim \text{Binomial}(16, p)$$

a random variable: # of babies choosing the helpful toy in a hypothetical replication of this experiment.

Could be different if:
  - different babies included in study
  - babies in a different mood
  - etc.

- Maximum likelihood <u>estimate</u> is $\hat{\Theta}^{MLE} = \frac{14}{16}$
  - estimate of p based on this specific sample

- Maximum likelihood <u>estimator</u> is $\hat{\Theta}^{MLE} = \frac{X}{n}$
  - a <u>random variable</u>: if we replicate the experiment we may observe a different value of X and in turn get a different estimate.

<u>Def.:</u> Suppose $\hat{\Theta}$ is an estimator of a parameter Θ. The probability distribution of the random variable $\hat{\Theta}$ is referred to as its <u>sampling distribution</u>

Recall that for a discrete random variable Y, the distribution is given by its p.m.f. : $f_Y(y) = P(Y=y)$.

For our Binomial example, the possible values of the estimator $\hat{\Theta}^{MLE}$ are $\frac{0}{16}, \frac{1}{16}, \frac{2}{16}, \dots, \frac{16}{16}$

Also, $\underbrace{P(\hat{\Theta}^{MLE} = \tfrac{14}{16}) = P(X=14)}_{\text{the only way the estimate would be } \frac{14}{16} \text{ is if we observe } x=14} = \binom{16}{14}\theta^{14}(1-\theta)^{16-14}$

A table to summarize the distribution of $\hat{\Theta}^{MLE}$:

~~$\hat{\Theta}^{MLE}$ | 0/16 | 1/16 | 2/16 | ...~~

~~$P(\hat{\Theta}^{MLE} = \hat{\theta}^{MLE})$ | $\binom{16}{0}\theta(1-\theta)$ | 16~~

$\hat{\Theta}^{MLE}$	$0/16$	$1/16$	$\dots$	$16/16$
$P(\hat{\Theta}^{MLE} = \hat{\theta}^{MLE} \mid \theta)$	$\binom{16}{0}\theta^0(1-\theta)^{16}$	$\binom{16}{1}\theta^1(1-\theta)^{16-1}$	$\dots$	$\binom{16}{16}\theta^{16}(1-\theta)^{16-16}$

Questions we might ask about the sampling distribution:

1) Is $E[\hat{\Theta}] = \theta$? (On average across all samples, will we recover the correct value of θ?)

<u>Def:</u> If $E[\hat{\Theta}] = \theta$, $\hat{\Theta}$ is said to be an <u>unbiased</u> estimator of θ.

<u>Def:</u> The <u>bias</u> of $\hat{\Theta}$ as an estimator of θ is

$\text{Bias}(\hat{\Theta}) = E(\hat{\Theta}) - \theta$. We want bias close to 0

$E(\hat{\Theta}^{MLE}) = E(\frac{X}{n}) = \frac{1}{n} \cdot E(X) = \frac{1}{n} \cdot n\theta = \theta$. For binomial model, $\hat{\Theta}^{MLE}$ is unbiased.

2) What is $Var(\hat{\Theta})$?

Describes variability in estimates of θ across different samples.

→ higher variance means estimates may be very different across different samples.

In general we want variance close to 0.

$Var(\hat{\Theta}^{MLE}) = Var(\frac{X}{n}) = \frac{1}{n^2} \cdot Var(X) = \frac{1}{n^2} n \cdot \cancel{p} \cdot (1-\cancel{p}) = \frac{\theta(1-\theta)}{n}$

Def: The standard error of an estimator is its standard deviation.

3) What is the average squared difference between $\hat{\Theta}$ and θ?

$$
\begin{aligned}
MSE(\hat{\Theta}) &= E[(\hat{\Theta} - \theta)^2] \\
&= \cancel{E[\hat{\Theta}^2 - 2\hat{\Theta}\theta + \theta^2]} \; E[(\hat{\Theta} - E(\hat{\Theta}) + E(\hat{\Theta}) - \theta)^2] \\
&= E[\{\hat{\Theta} - E(\hat{\Theta})\}^2 + 2(\hat{\Theta} - E(\hat{\Theta}))\underbrace{(E(\hat{\Theta}) - \theta)}_{\text{nothing random}} + (E(\hat{\Theta}) - \theta)^2] \\
&= E[\{\hat{\Theta} - E(\hat{\Theta})\}^2] + 2(E(\hat{\Theta}) - \theta) \cdot E\{\hat{\Theta} - E(\hat{\Theta})\} \quad \leftarrow \text{not random} \\
&\quad + E[\underbrace{\{E(\hat{\Theta}) - \theta\}}_{\text{nothing random}}^2] \\
&= E[\{\hat{\Theta} - E(\hat{\Theta})\}^2] + 2\{E(\hat{\Theta}) - \theta\} \cdot \overset{0}{\cancel{\{E(\hat{\Theta}) - E(\hat{\Theta})\}}} \\
&\quad + \{E(\hat{\Theta}) - \theta\}^2 \\
&= Var(\hat{\Theta}) + \{Bias(\hat{\Theta})\}^2
\end{aligned}
$$

We would like $MSE(\hat{\Theta})$ to be small.

Ex: $MSE(\hat{\Theta}) = Var(\hat{\Theta}) + \{Bias(\hat{\Theta})\}^2$

$$= \frac{\theta(1-\theta)}{n} + 0^2 = \frac{\theta(1-\theta)}{n}$$