Method of Moments:

Remember definition of a moment:

Suppose X is a random variable.
The k'th moment of the distribution of X is

$$\mu_k = E[X^k]$$

The k'th <u>sample moment</u> based on a sample $X_1, \ldots, X_n$ is

$$\hat{\mu}_k = \frac{1}{n} \sum_{i=1}^{n} X_i^k$$

Method of moments: set the first few sample moments equal to the corresponding moments of the distribution whose parameters we want to estimate; solve the resulting system of equations for unknown parameters

Notes:

- Typically need as many moments as there are unknown parameters
- With 1 unknown parameter, this amounts to setting the theoretical mean equal to the sample mean
- With 2 parameters, equate means and variances (but with a divisor of n for sample variance)

Example 1: Method of Moments for Normal Distribution

If $X \sim \text{Normal}(\mu, \sigma^2)$ then:

2 parameters, so consider first 2 moments, μ_1 and μ_2:

$\mu_1 = E(X) = \mu$

$$Var(X) = \sigma^2 = E(X^2) - \{E(X)\}^2$$

$$\Rightarrow \mu_2 = E(X^2) = \{E(X)\}^2 + Var(X)$$

$$= \mu^2 + \sigma^2$$

Set these expressions equal to corresponding sample moments, then solve for μ and σ^2:

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \mu_1 = \mu \qquad (\text{Eq } 1)$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \mu_2 = \mu^2 + \sigma^2 \qquad (\text{Eq } 2)$$

From Eq 1, $\hat{\mu}^{MoM} = \frac{1}{n}\sum_{i=1}^{n} X_i$

From Eq 2, $\hat{\sigma}^{2\,MoM} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\hat{\mu}^{MoM}\right)^2$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2$$

Last equation is true since

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i^2 - 2X_i\bar{X} + \bar{X})^2 \quad \cancel{\frac{1}{n}\sum X_i^2 -}$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n}\cdot 2\bar{X}\cdot\sum_{i=1}^{n} X_i + \frac{1}{n}\sum_{i=1}^{n}\bar{X}^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - 2\bar{X}^2 + \frac{1}{\cancel{n}}\cancel{n}\,\bar{X}^2$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

Example 2: MoM for Gamma Distribution

If $X \sim \text{Gamma}(\alpha, \lambda)$ then

$$E(X) = \frac{\alpha}{\lambda}, \quad Var(X) = \frac{\alpha}{\lambda^2}$$

since $Var(X) = E(X^2) - \{E(X)\}^2$

$$E(X^2) = Var(X) + \{E(X)\}^2 = \frac{\alpha}{\lambda^2} + \left(\frac{\alpha}{\lambda}\right)^2$$

$$\bar{X} = E(X) = \frac{\alpha}{\lambda} \Rightarrow \alpha = \lambda \bar{X}$$

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 = \frac{\alpha}{\lambda^2} + \left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{\cancel{\lambda}\bar{X}}{\lambda^{\cancel{2}}} + (\bar{X})^2$$

$$\Rightarrow \lambda\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right) = \bar{X}$$

$$\Rightarrow \lambda = \frac{\bar{X}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2} = \frac{\bar{X}}{\hat{\sigma}^2}$$