Stat 343: Motivation for Method of Moments and Numerical Optimization of the Likelihood

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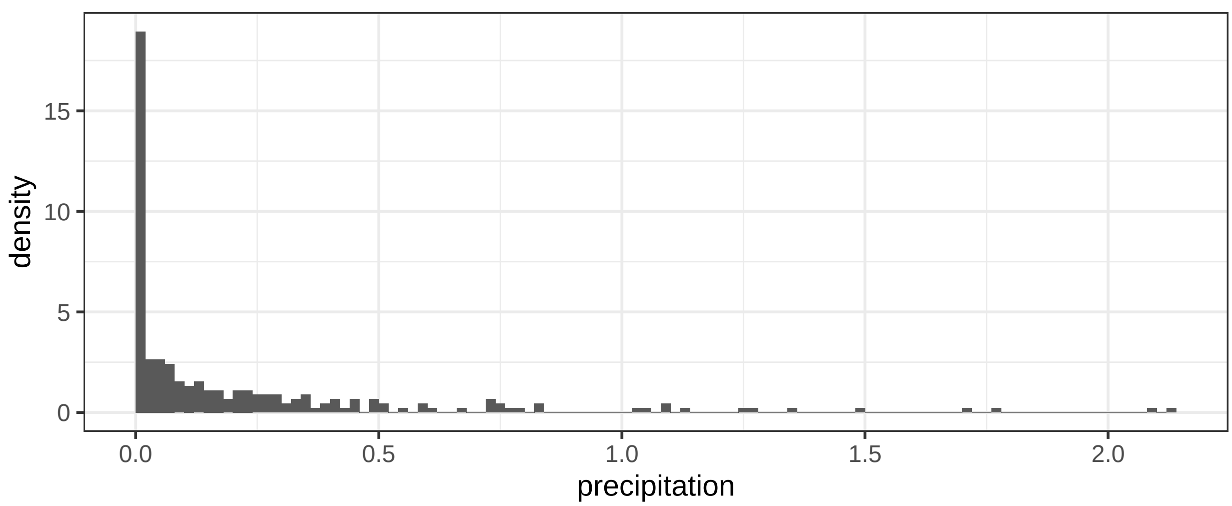
Rainfall in Illinois, all storms from 1960 - 1964

The code below reads in the data and makes an initial plot:

```
library(tidyverse)

# Precipitation in Illinois
# Amount of rainfall in 272 storms from 1960 to 1962
il_storms <- bind_rows(
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois60.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois61.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois62.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois63.txt",
    col_names = FALSE),
  read_csv("http://www.evanlray.com/data/rice/Chapter%2010/illinois64.txt",
    col_names = FALSE)
)
names(il_storms) <- "precipitation"

ggplot(data = il_storms, mapping = aes(x = precipitation)) +
  geom_histogram(boundary = 0, binwidth = 0.02, mapping = aes(y = ..density..)) +
  theme_bw()
```



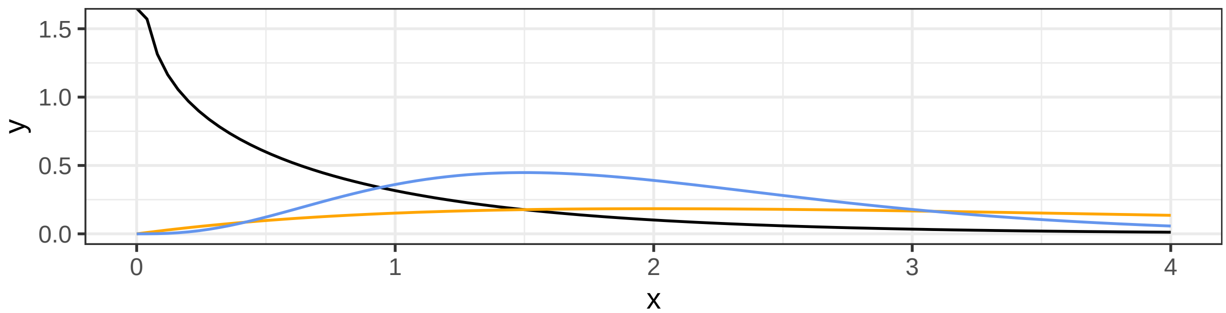
Let's model these data with a Gamma distribution and estimate the parameters by maximum likelihood

Gamma(α, λ) Distribution (copied from the "Common Distributions" handout)

A non-negative real number. Be careful – there are multiple other parameterizations used in other sources.

parameters	$\alpha \geq 0$: shape parameter, $\lambda > 0$: rate parameter
p.f.	$f(x\|\alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$
Mean	$\frac{\alpha}{\lambda}$
Variance	$\frac{\alpha}{\lambda^2}$
R functions	`dgamma(..., shape =` α`, rate =` λ`), pgamma, qgamma, rgamma`

```
ggplot(data = data.frame(x = c(0, 4)), mapping = aes(x = x)) +
  stat_function(fun = dgamma, args = list(shape = 0.8, rate = 1), color = "black") +
  stat_function(fun = dgamma, args = list(shape = 2, rate = 0.5), color = "orange") +
  stat_function(fun = dgamma, args = list(shape = 4, rate = 2), color = "cornflowerblue") +
  theme_bw()
```

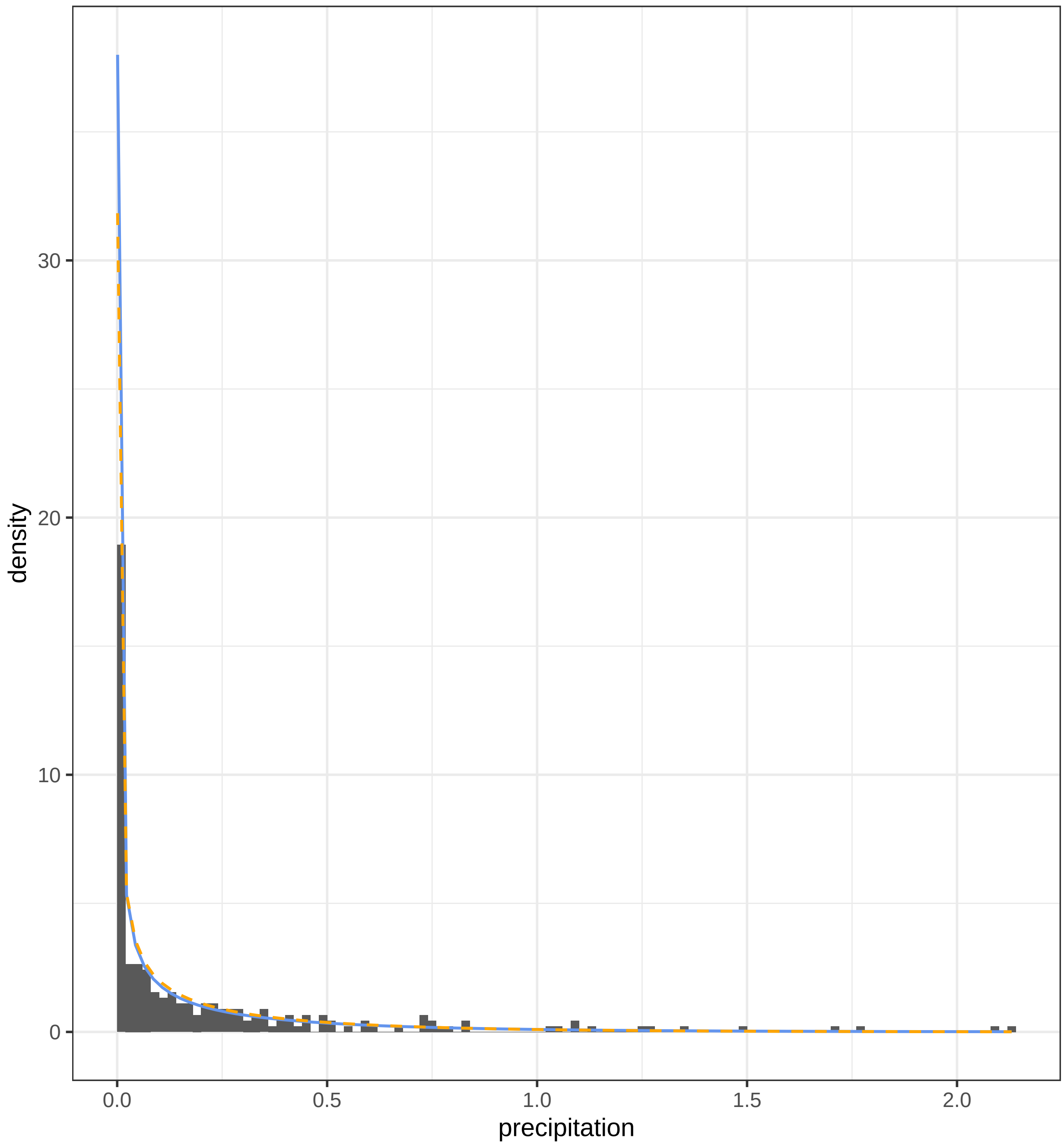


Preview

For our data set:

- The method of moments estimates are $\hat{\alpha}^{MoM} = 0.3779$ and $\hat{\beta}^{MoM} = 1.6840$
- The maximum likelihood estimates are about $\hat{\alpha}^{MLE} = 0.4408$ and $\hat{\beta}^{MLE} = 1.9644$

```
ggplot(data = il_storms, mapping = aes(x = precipitation)) +
  geom_histogram(boundary = 0, binwidth = 0.02, mapping = aes(y = ..density..)) +
  stat_function(fun = dgamma, args = list(shape = 0.3779, rate = 1.6840), color = "cornflowerblue", size =
  stat_function(fun = dgamma, args = list(shape = 0.4408, rate = 1.9644), color = "orange", linetype = 2, s
  theme_bw()
```



Attempt at Maximum Likelihood for a Gamma(α, β)

Suppose we model $X_1, \ldots, X_n$ as i.i.d. with each $X_i \sim \text{Gamma}(\alpha, \beta)$.

The likelihood function is

$$
\begin{aligned}
\mathcal{L}(\alpha, \beta | x_1, \ldots, x_n) &= \prod_{i=1}^{n} \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \\
&= \left\{ \frac{\lambda^\alpha}{\Gamma(\alpha)} \right\}^n \left\{ \prod_{i=1}^{n} x_i^{\alpha-1} \right\} e^{-\lambda \sum_{i=1}^{n} x_i}
\end{aligned}
$$

The log-likelihood function is

$$
\ell(\alpha, \beta | x_1, \ldots, x_n) = n\alpha \log(\lambda) - n \log \{\Gamma(\alpha)\} + \sum_{i=1}^{n} (\alpha - 1) \log(x_i) - \lambda \sum_{i=1}^{n} x_i
$$

The partial derivatives are

$$
\frac{\partial}{\partial \alpha} \ell(\alpha, \beta | x_1, \ldots, x_n) = n \log(\lambda) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log(x_i) \tag{1}
$$

$$
\frac{\partial}{\partial \beta} \ell(\alpha, \beta | x_1, \ldots, x_n) = \frac{n\alpha}{\lambda} - \sum_{i=1}^{n} x_i \tag{2}
$$

Setting Equation (2) equal to 0 and solving for λ, we obtain

$$
\lambda = \frac{n\alpha}{\sum_{i=1}^{n} x_i} = \frac{\alpha}{\bar{x}} \tag{3}
$$

Substituting Equation (3) into Equation (1) and setting equal to 0, we obtain

$$
n \log(\alpha) - n \log(\bar{x}) + \sum_{i=1}^{n} \log(x_i) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = 0 \tag{4}
$$

However, there is no way to rearrange Equation (4) to solve for α! We are stuck!!!

Alternatives

- Method of Moments: an entirely different strategy that is sometimes equivalent to maximum likelihood. When they are different, maximum likelihood is superior (we will discuss this later), but in practice the differences may be small.
- Numeric optimization of the likelihood function