# $\chi^2$, t, and F distributions



Why do we care?

These distributions come up a lot in building confidence intervals and conducting hypothesis tests.

Ex: $X_1, \ldots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$

Estimate $\mu$ by $\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} X_i$

C.I. and hypothesis test are based on

$$\frac{\bar{X} - \mu}{\sqrt{\frac{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}{n}}} \sim t_{n-1}$$

- what is a $t_{n-1}$ distribution?
- why / how do we know that

$$\frac{\bar{X} - \mu}{\sqrt{\frac{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}{n}}}$$

follows a t distribution with n - 1 degrees of freedom?

# $\chi^2$ distribution

- building block for t and F distributions
- we will see, also occasionally useful on its own

- If ~~Z is~~ $Z \sim \text{Normal}(0,1)$, ~~then~~ and we define a new random variable $U = Z^2$, then

$$U \sim \chi^2_1 \quad (\chi^2 \text{ with 1 degree of freedom})$$

Ex: If $X \sim \text{Normal}(0, \sigma^2)$ then $\frac{X-\mu}{\sigma} \sim \text{Normal}(0,1)$, so $\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2_1$

- If $U_1, U_2, \ldots, U_n$ are iid (independent and identically distributed) random variables with a $\chi^2_1$ distribution, ~~then~~ and we define $V = U_1 + U_2 + \cdots + U_n$, then $V \sim \chi^2_n$

Ex: If $X_1, \ldots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$ then

$$\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_n$$

## t Distribution

If $Z \sim \text{Normal}(0,1)$, $U \sim \chi^2_n$, and $Z$ and $U$ are independent

then $$\frac{Z}{\sqrt{U/n}} \sim t_n$$

## F Distribution

Let $U$ and $V$ be independent $\chi^2$ random variables with $m$ and $n$ degrees of freedom respectively.

Then $$\left(\frac{U/m}{V/n}\right) \sim F_{m,n}$$

## Ex.:

Suppose $T \sim t_n$. Define $X = T^2$

What is the distribution of $X$?

# Sample Mean and Variance, Normal Dist'n

Suppose $X_1, \ldots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$

Consider 2 new random variables

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

Claim: $\bar{X} \sim \text{Normal}(\mu, \sigma^2/n)$

Claim: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

Claim: $\bar{X}$ and $S^2$ are independent,

so $\bar{X}$ and $\frac{(n-1)S^2}{\sigma^2}$ are independent

All 3 claims above can be proved using moment generating functions, see the textbook.

Claim: $\dfrac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$

Verification: divide numerator & denominator by $\sigma/\sqrt{n}$:

$$\frac{(\bar{X}-\mu)/(\sigma/\sqrt{n})}{S/\sqrt{n}\,/(\sigma/\sqrt{n})} = \frac{\left[\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right]}{\frac{(n-1)S}{\sigma}/(n-1)}$$

= ratio of Normal (0,1) r.v.
and $\chi^2_{n-1}$ r.v. divided by its d.f.

↑

(this is the def. of a $\chi^2_{n-1}$ r.v.)