Probability Practice Examples

Example 1

I toss a fair coin three times and record whether each toss resulted in heads or tails. For your reference, here is an enumeration of the sample space for this experiment, where htt represents the outcome that a heads came up on the first toss, and tails on the second and third tosses:

$\{hhh, hht, hth, htt, thh, tht, tth, ttt\}$

Define the following random variables:

X = the number of heads on the first toss

Y = the total number of heads across all three tosses

(a) By counting outcomes in the sample space, fill in the table below to specify the marginal distribution of Y. (A number inside the table should be a value of $f_Y(y)$.)

	y = 0	y = 1	y = 2	y = 3
$f_Y(y)$	1/8	3/8	3/8	1/8

(b) By counting outcomes in the sample space, fill in the table below to specify the joint p.m.f. of (X, Y). (A number inside the table should be a value of $f_{X,Y}(x, y)$.)

	y = 0	y = 1	y = 2	y = 3
x = 0	1/8	2/8	1/8	0/8
x = 1	0/8	1/8	2/8	1/8

(c) By counting outcomes in the sample space, fill in the table below to specify the conditional p.m.f. of X given that $Y = 1$. (A number inside the table should be a value of $f_{X|Y}(x|Y = 1)$.)

Consider only these outcomes in the sample space, where $Y = 1$:
htt, tht, tth

	$f_{X\|Y}(x\|Y = 1)$
x = 0	2/3
x = 1	1/3

(d) Verify that $f_{X|Y}(0|Y = 1) = \frac{f_{X,Y}(0,1)}{f_Y(1)}$, and similar for $f_{X|Y}(1|Y = 1)$

$$f_{X|Y}(0|Y=1) = 2/3 = \frac{2/8}{3/8} = \frac{f_{X,Y}(0,1)}{f_Y(1)} \checkmark$$

$$f_{X|Y}(1|Y=1) = 1/3 = \frac{1/8}{3/8} = \frac{f_{X,Y}(1,1)}{f_Y(1)} \checkmark$$

Example 2

The marginal distribution of X and the conditional distribution of Y given X are as follows:

$$\begin{aligned} f_X(x) &= e^{-x} && 0 < x < \infty \\ f_{Y|X=x}(y|X=x) &= e^{-(y-x)} && 0 < x < y < \infty \end{aligned}$$

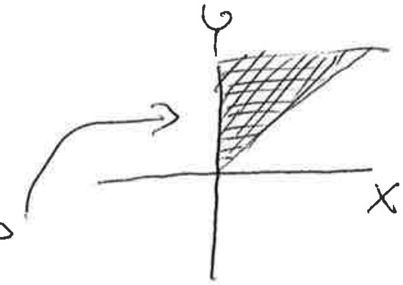
(a) Find the joint distribution of X and Y, $f_{X,Y}(x,y)$.

$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x) = e^{-x} \cdot e^{-(y-x)}$

$= e^{-x-y+x}$

$= e^{-y}$

on the support $0 < x < y < \infty$



(b) Find the marginal distribution of Y, $f_Y(y)$.

$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx = \int_0^y e^{-y}\,dx = e^{-y} \cdot x \Big|_{x=0}^{y}$

$= e^{-y}(y-0) = ye^{-y}$

On the support $0 < y < \infty$

(c) Why doesn't $f_{X,Y}(x,y) = f_X(x)f_Y(y)$? (Under what condition would $f_{X,Y}(x,y) = f_X(x)f_Y(y)$, and what does the fact that this equation doesn't hold mean about X and Y?)

The random variables X and Y are not independent.

$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$ only if X and Y are independent.

(d) Find the conditional distribution of X given that $Y = y$.

on the support

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-y}}{ye^{-y}} = \frac{1}{y}, \quad 0 < x < y$$

Example 3

The continuous random variables X and Y have a joint pdf with the following form:

$$f_{X,Y}(x,y) = c(x+y)^2, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1$$

Find the value of c.

We must have $1 = \int_0^1 \int_0^1 f_{X,Y}(x,y)\,dx\,dy = c \int_0^1 \int_0^1 (x^2 + 2xy + y^2)\,dx\,dy$

$$= c \cdot \int_0^1 \left(x^2 y + xy^2 + \frac{1}{3}y^3 \Big|_{y=0}^{1} \right) dx = c \cdot \int_0^1 x^2 + x + \frac{1}{3}\,dx$$

$$= c \cdot \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x \Big|_{x=0}^{1} \right) = c \cdot \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3} \right)$$

$$\Rightarrow 1 = c \cdot \frac{7}{6} \Rightarrow c = \frac{6}{7}$$

Example 4

Suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$, and define $Z = \frac{X-\mu}{\sigma}$.

Show that $E(Z) = 0$ and $Var(Z) = 1$.

$$E[Z] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma} \cdot E[X - \mu] = \frac{1}{\sigma}\left(E(X) - \mu\right)$$

$$= \frac{1}{\sigma}(\mu - \mu) = 0$$

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} \cdot Var(X) = \frac{1}{\sigma^2} \cdot Var(X)$$

$$= \frac{1}{\sigma^2} \cdot \sigma^2 = 1.$$