# Calculus for Stat 343

## Pre-Calculus

### Sums

You can factor anything that doesn't depend on the summation index out of a sum:

$$\sum_{i=1}^{n} cx_i = (cx_1 + cx_2 + \cdots + cx_n) = c(x_1 + x_2 + \cdots + x_n) = c\sum_{i=1}^{n} x_i$$

### Products

You can factor anything that doesn't depend on the product index out of a product, but you have to raise it to the power of the number of terms in the product:

$$\prod_{i=1}^{n} cx_i = (cx_1)(cx_2)\cdots(cx_n) = c^n \prod_{i=1}^{n} x_i$$

### Logarithms and Exponents

$a$, $b$, and $c$ are real numbers, $e \approx 2.718281828459$ is Euler's number.

$\log(a)$ is defined to be the number to which you raise $e$ in order to get $a$: $e^{\log(a)} = a$.

$\log(e) = 1$

$\log(ab) = \log(a) + \log(b)$

$\log(a^b) = b\log(a)$

$\log(a/b) = \log(a) - \log(b)$

$a^b \cdot a^c = a^{b+c}$

### Gamma Function

The Gamma ($\Gamma$) function is basically a continuous version of the factorial. If $a$ is an integer, then

$\Gamma(a) = (a-1)!$

If $a$ is a real number, the $\Gamma$ function is still defined, and it's basically a smooth interpolation between the factorials of nearby integers. There's a way to define the $\Gamma$ function as an integral, but we won't need to know that.

# Differential Calculus in One Variable

### Chain rule:

Suppose $f$ and $g$ are functions, and define $h$ by $h(x) = f(g(x))$. Then $h'(x) = f'(g(x)) \cdot g'(x)$

### Derivative of a polynomial:

If $a \neq 0$ then $\frac{d}{dx}x^a = ax^{a-1}$

Two special cases that come up a lot are $a = -1$ and $a = -2$:

If $a = -1$ then $\frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -1x^{-2} = \frac{-1}{x^2}$

If $a = -2$ then $\frac{d}{dx}\frac{1}{x^2} = \frac{d}{dx}x^{-2} = -2x^{-3} = \frac{-2}{x^3}$

### Derivative of an exponential:

$\frac{d}{dx}e^x = e^x$

In combination with the chain rule, we get

$\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$

### Derivative of a logarithm:

$\frac{d}{dx}\log(x) = \frac{1}{x}$

In combination with the chain rule, we get

$\frac{d}{dx}\log(f(x)) = \frac{1}{f(x)}f'(x)$

### Finding maximum and minimum of a function

To find a maximum or minimum of a function $f(x)$, we can often use this procedure:

1. Find a **critical point** $x^*$ by setting the first derivative to 0 and solving for $x$.
2. Verify that the critical point is a maximum or minimum; in this class, we will typically use the second derivative test to do this:
    1. If $f''(x^*) > 0$ (at the critical point), the critical point is a **local minimum** of $f$
    2. If $f''(x) > 0$ (at all values of $x$), the critical point is a **global minimum** of $f$
    3. If $f''(x^*) < 0$ (at the critical point), the critical point is a **local maximum** of $f$
    4. If $f''(x) < 0$ (at all values of $x$), the critical point is a **global maximum** of $f$

Let's illustrate by finding an extreme point of the function $f(x) = 3x^2 - 12x + 14$ and seeing whether it is a local or global minimum or maximum.

Step 1: Find a critical point

$$\frac{d}{dx}f(x) = \frac{d}{dx}3x^2 - 12x + 14 = 6x - 12 = 0$$

Solving for $x$, we find that $x^* = 2$ is a critical point.

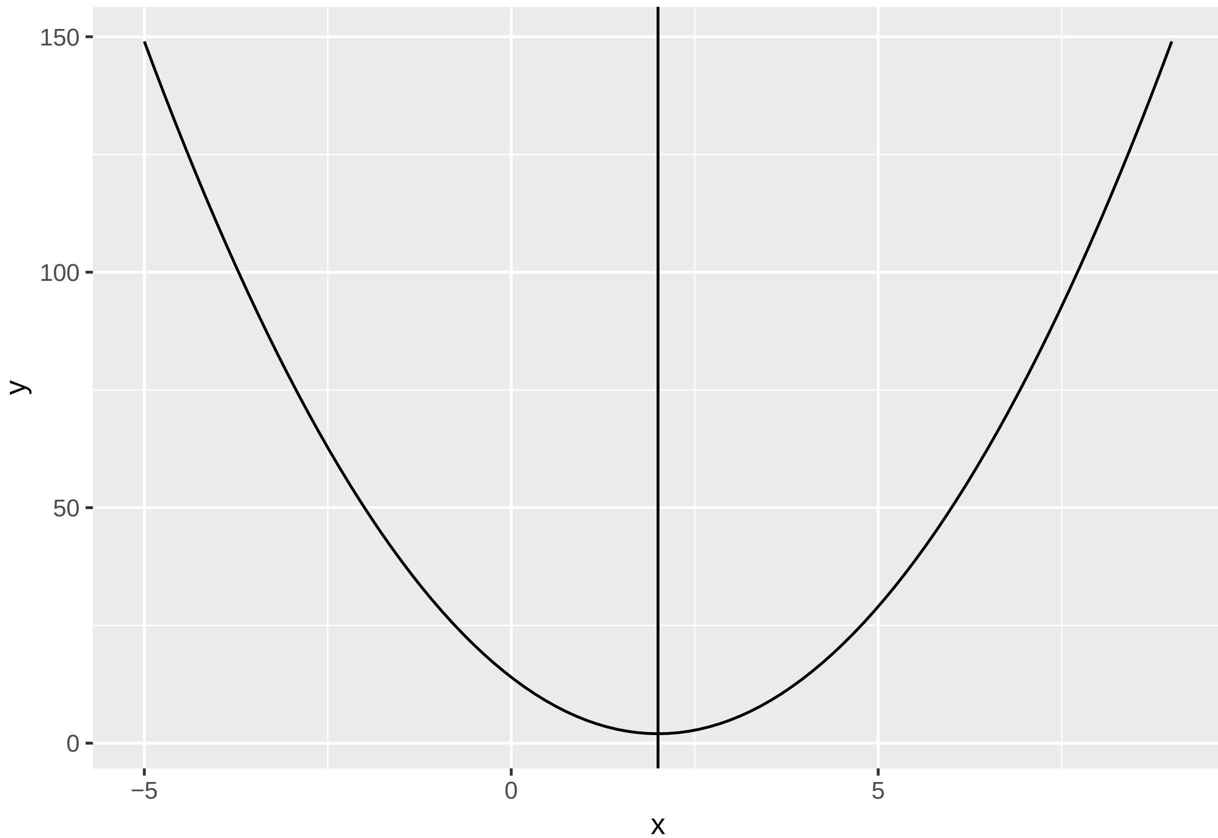
Step 2: Determine whether the critical point is a maximum or minimum, and whether it is local or global

$$\frac{d^2}{dx^2}f(x) = \frac{d^2}{dx^2}3x^2 - 12x + 14 = \frac{d}{dx}6x - 12 = 6$$

Since the second derivative is positive for all values of $x$, the critical point $x^* = 2$ is a global minimum of $f$. Here's a picture:

```r
library(ggplot2)
f <- function(x) {
  3*x^2 - 12*x + 14
}

ggplot(data = data.frame(x = c(-5, 9)), mapping = aes(x = x)) +
  stat_function(fun = f) +
  geom_vline(xintercept = 2)
```




**Taylor's Theorem**

I adapted this statement of Taylor's theorem from Wikipedia: https://en.wikipedia.org/wiki/Taylor%27s_theorem#Taylor's_theorem_in_one_real_variable

Let $k \geq 1$ be an integer and suppose that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ is $k$ times differentiable at the point $a \in \mathbb{R}$. Define the $k$-th order polynomial approximation to $f$ centered at $a$ by

$P_k(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x - a)^k$

Then there exists a function $h_k : \mathbb{R} \rightarrow \mathbb{R}$ such that:

- $f(x) = P_k(x) + h_k(x)(x - a)^k$ and
- $\lim_{x \rightarrow a} h_k(x) = 0$

(You can get more specific about what the function $h_k$ looks like and rates of convergence to 0, but we don't need to do that.)

The main points are:

1. For values of $x$ near $a$, the function $f(x)$ can be well approximated by a polynomial, and the polynomial's coefficients can be obtained by the derivatives of $f$.
2. The approximation is better if you use a higher degree polynomial.

As an example, let's approximate $f(x) = e^{5x}$ by a second-order Taylor polynomial centered at $a = 1$. We will need the first and second derivatives of $f(x)$:

$$\frac{d}{dx}e^{5x} = e^{5x} \cdot 5$$
$$\frac{d^2}{dx^2}e^{5x} = \frac{d}{dx}e^{5x} \cdot 5 = e^{5x} \cdot 25$$

$$\begin{aligned} P_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 \\ &= e^{5\cdot 1} + 5e^{5\cdot 1}(x-1) + \frac{25e^{5\cdot 1}}{2}(x-1)^2 \end{aligned}$$

The claim is that $f(x)$ looks very similar to $P_2(x)$ for values of $x$ near $a = 1$. Let's verify with a picture:

```
library(ggplot2)
f <- function(x) {
  exp(5 * x)
}

P_2 <- function(x) {
  exp(5) + 5 * exp(5) * (x - 1) + (25 * exp(5)) / 2 * (x - 1)^2
}

temp_df <- data.frame(x = c(0.5, 1.5))
ggplot(data = temp_df, mapping = aes(x =x)) +
  stat_function(fun = f) +
  stat_function(fun = P_2, color = "orange")
```