Suppose $X_1, \dots, X_n \overset{iid}{\sim} \text{Normal}(\mu, \sigma^2)$,

both μ and σ^2 unknown

$\underset{\sim}{\theta} = (\mu, \sigma^2)$

$\Omega = \{(\mu, \sigma^2) : \mu \in \mathbb{R}, \sigma^2 \geq 0\}$

Consider a test of

$H_0 : \mu = \mu_0 \quad \rightarrow \quad H_0 : \underset{\sim}{\theta} \in \Omega_0 = \{(\mu, \sigma^2) : \mu = \mu_0, \sigma^2 \geq 0\}$

$H_A : \mu \neq \mu_0 \quad\quad\quad\; H_A : \underset{\sim}{\theta} \in \Omega_1 = \{(\mu, \sigma^2) : \mu \neq \mu_0, \sigma^2 \geq 0\}$

$$W = \frac{\max_{\theta \in \Omega_0} \mathcal{L}(\theta \mid \underset{\sim}{x})}{\max_{\theta \in \Omega} \mathcal{L}(\theta \mid \underset{\sim}{x})}$$

For denominator: we have previously shown the MLE's of μ and σ^2 are

$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2$

For numerator: We know $\mu = \mu_0$.

Given that, can show the MLE of $\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu_0)^2$

For a particular μ and σ^2, joint pdf of $X_1, \dots, X_n$ is

$$\mathcal{L}(\mu, \sigma^2 \mid \underset{\sim}{x}) = f_{X_1, \dots, X_n \mid \mu, \sigma^2}(x_1, \dots, x_n) = \prod_{i=1}^{n} f_{X_i \mid \mu, \sigma^2}(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{\frac{-1}{2\sigma^2}(x_i - \mu)^2\right\}$$

$$= (2\pi)^{-n/2} (\sigma^2)^{\frac{-n}{2}} \exp\left\{\frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right\}$$

The p-value is

$P(W \leq w \mid H_0 \text{ correct})$

a constant. to avoid confusion, rename as c.

$$W = \frac{\mathcal{L}\left(\mu_0, \frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2\right)}{\mathcal{L}\left(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)}$$

$$= \frac{\cancel{(2\pi)^{-n/2}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2\right)^{-n/2}\cancel{\exp\left\{\frac{-1}{2\cdot\frac{1}{n}\sum(X_i-\mu_0)^2}\sum_{i=1}^{n}(X_i-\mu_0)^2\right\}}}{\cancel{(2\pi)^{-n/2}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)^{-n/2}\cancel{\exp\left\{\frac{-1}{2\cdot\frac{1}{n}\sum(X_i-\bar{X})^2}\sum_{i=1}^{n}(X_i-\bar{X})^2\right\}}}$$

p-value $= P(W \le c) = P\left(\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sum_{i=1}^{n}(X_i-\mu_0)^2}\right)^{n/2} \le c\right)$

All probabilities have should be given H_0 is correct, $\mu=\mu_0$.

$$= P\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{\sum_{i=1}^{n}(X_i-\mu_0)^2} \le c_1\right), \quad c_1 = c^{2/n}$$

$$= P\left(\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu_0)^2}{\sum_{i=1}^{n}(X_i-\bar{X})^2} \ge c_2\right), \quad c_2 = \frac{1}{c_1}; \text{ note } \sum_{i=1}^{n}(X_i-\mu_0) = \sum_{i=1}^{n}(X_i-\bar{X})^2 + n(\bar{X}-\mu_0)^2$$

$$= P\left(1 + \frac{(\bar{X}-\mu_0)^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2} \ge c_2\right)$$

$$= P\left(\frac{(\bar{X}-\mu_0)^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2/(n-1)} \ge c_3\right), \quad c_3 = \frac{c_2-1}{(n-1)}$$

$$= P\left(\frac{|\bar{X}-\mu_0|}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}/\sqrt{n}} \ge c_4\right), \quad c_4 = \sqrt{c_3}$$

$$= P(t \le -c_4 \text{ or } t \ge c_4 \mid \mu=\mu_0),$$

where $t = \frac{\bar{X}-\mu_0}{s/\sqrt{n}}$, $\quad s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2}$,

Connections Between Confidence Intervals & Hypothesis Tests

Suppose that for every $\theta_0 \in \Omega$ there is a test of $H_0: \theta = \theta_0$ with $\underbrace{\text{size } \alpha}_{P(\text{Reject } H_0 \mid H_0 \text{ correct}) = \alpha}$.

Denote the rejection region for the test by $R(\theta_0)$. (we reject H_0 if $X \in R(\theta_0)$)

The set

$$C(\underset{\sim}{X}) = \{\theta \in \Omega : \underset{\sim}{X} \notin R(\theta_0)\}$$

is a $100(1-\alpha)\%$ confidence set for θ

in general, not guaranteed to be an interval.

<u>In words</u>: the set of values θ_0 for which we would <u>fail to reject</u>

$H_0: \theta = \theta_0$

is a $100(1-\alpha)\%$ CI for θ

(set)

<u>Proof</u>:

$$P(\theta \in C(\underset{\sim}{X}) \mid \theta)$$

$$= P(\underset{\sim}{X} \notin R(\theta) \mid \theta)$$ (from the way we defined $C(\underset{\sim}{X})$: $\theta \in C(\underset{\sim}{X}) \Leftrightarrow \underset{\sim}{X} \notin R(\theta)$)

$$= 1 - P(\underset{\sim}{X} \in R(\theta) \mid \theta)$$

$$= 1 - \alpha$$

~~Also suppose~~

Other direction: Suppose $C(\underset{\sim}{X})$ is a $100(1-\alpha)\%$ CI for θ.

Then a rejection region for a size α test of $H_0: \theta = \theta_0$ is

$$R(\theta_0) = \{\underset{\sim}{X} \mid \theta_0 \notin C(\underset{\sim}{X})\}$$

In words: Reject $H_0: \theta = \theta_0$ if θ_0 is not in the confidence interval for θ.

Confidence intervals from likelihood ratio tests:

For a test of $H_0: \theta = \theta_0$ vs $H_A: \theta \neq \theta_0$,

LRT has p-value $= P(W \leq w \mid \theta = \theta_0)$

Reject if p-value $\leq \alpha$.

$\Rightarrow$ Reject if W is small $(w \leq c)$

$\Rightarrow$ fail to reject if w is large $(w > c)$

$\Rightarrow$ θ is in our CI if the likelihood ratio at that value of θ is "big enough".