Bayesian Analysis of the Paint Drying Example

- Data Model: $X_1, \ldots, X_5 \overset{\text{i.i.d.}}{\sim} \text{Normal}(\theta, 5^2)$
- We know that θ is either 10 (quick dry) or 25 (regular).
- We don't know which value of theta is correct. Suppose we adopt a prior with equal probablity for each value of θ:

$f_\Theta(10) = f_\Theta(25) = 0.5$

- Here's some algebra that says the joint pdf of $X_1, \ldots, X_n$ is proportional to the pdf of $\bar{X}$ (based on the data following a Normal(θ, σ^2) distribution):

$$
\begin{aligned}
f_{X_1,\ldots,X_n|\Theta}(x_1, \ldots, x_n|\theta) &= \prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-1}{2\sigma^2}(x_i - \theta)^2\right] \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta)^2\right] \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x} + \bar{x} - \theta)^2\right] \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}\left\{(x_i - \bar{x})^2 + 2(x_i - \bar{x})(\bar{x} - \theta) + (\bar{x} - \theta)^2\right\}\right] \\
&= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2\right] \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}2(x_i - \bar{x})(\bar{x} - \theta)\right] \exp\left[\frac{-1}{2\sigma^2}\sum_{i=1}^{n}(\bar{x} - \theta)^2\right] \\
&= c \exp\left[\frac{-1}{2\sigma^2}2(\bar{x} - \theta)(n\bar{x} - n\bar{x})\right] \exp\left[\frac{-1}{2\sigma^2}n(\bar{x} - \theta)^2\right] \\
&= c \exp\left[\frac{-1}{2\sigma^2/n}(\bar{x} - \theta)^2\right] \\
&= c f_{\bar{X}|\Theta}(\bar{x}|\theta)
\end{aligned}
$$

- Where $\bar{X} \sim \text{Normal}(\theta, \sigma^2/n)$

Find the posterior probability that $\Theta = 25$ in terms of the pdf of $\bar{X}$ based on $\theta = 10$ and $\theta = 25$, $f_{\bar{X}|\Theta}(20|10)$ and $f_{\bar{X}|\Theta}(20|25)$.

Posterior probabilities if the two options are $\theta = 25$ and $\theta = 10$:

```
dnorm(20, mean = 25, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 10, sd = 5))
```

```
## [1] 0.8175745
```

```
dnorm(20, mean = 10, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 10, sd = 5))
```

```
## [1] 0.1824255
```

Posterior probabilities if the two options are $\theta = 25$ and $\theta = 0$:

```
dnorm(20, mean = 25, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 0, sd = 5))
```

```
## [1] 0.9994472
```

```
dnorm(20, mean = 0, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 0, sd = 5))
```

```
## [1] 0.0005527786
```

Posterior probabilities if the two options are $\theta = 25$ and $\theta = 20$:

```
dnorm(20, mean = 25, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 20, sd = 5))
```

```
## [1] 0.3775407
```

```
dnorm(20, mean = 20, sd = 5) / (dnorm(20, mean = 25, sd = 5) + dnorm(20, mean = 20, sd = 5))
```

```
## [1] 0.6224593
```