Results for Bayesian inference over the last 2 weeks:

- If a conjugate prior is used, we can get exact posterior results "easily"
- If n is large and conditions are satisfied, the posterior is approximately $\Theta \mid X_1,\dots,X_n \sim \text{Normal}\left(\hat{\theta}^{MLE}, \dfrac{1}{-\frac{d^2}{d\theta^2} l(\theta \mid x_1,\dots,x_n)\big|_{\theta=\hat{\theta}^{MLE}}}\right)$
- We can always approximate the posterior by a sample from it (MCMC),

$$\Theta_1,\dots,\Theta_m \sim f_{\Theta \mid X_1,\dots,X_n}(\theta \mid x_1,\dots,x_n)$$

We know how to find point estimators by MLE.

Next few weeks:

- Exact confidence intervals (rarely useful, ~~[illegible]~~)
- Approximate, from large sample normal approximations
- Approximate, using ~~re~~sampling to approximate sampling distribution (Bootstrap)

Also, will ~~see~~ see large sample properties of MLE's.

Reminder of set-up for ~~[illegible]~~ confidence intervals:

A $(1-\alpha)*100\%$ confidence interval is a pair of random variables A, B such that $P(A \le \theta \le B) = 1-\alpha$.

Once we observe data, we get a <u>realized confidence interval</u> $[a,b]$.

We cannot make probability statements about $[a,b]$.

<u>Coverage Probability</u>: The probability that the interval contains the parameter being estimated: $P(A \le \theta \le B) = \boxed{1-\alpha}$

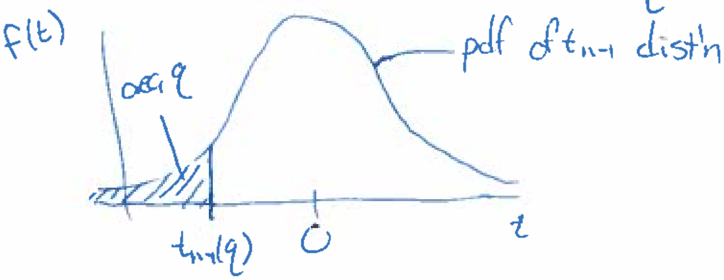
<u>Nominal Coverage Probability</u>: The claimed coverage probability for a confidence interval, which may not be correct if assumptions made are not met, or approximations are not accurate.

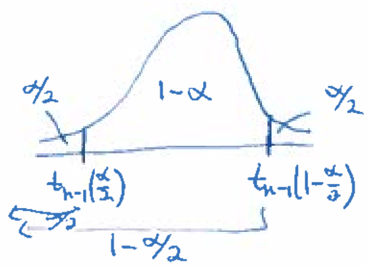
Exact interval estimates for μ in a normal model: (Example A, Section 8.5.3)

Suppose $X_1, \ldots, X_n \sim \text{Normal}(\mu, \sigma^2)$

Then $$\frac{\sqrt{n}(\bar{X}-\mu)}{S} \sim t_{n-1}, \quad \text{where } S = \left\{ \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right\}^{1/2}$$

Denote by $t_{n-1}(q)$ the q'th quantile of the t_{n-1} distribution
(note: consistent with r's qt function, not consistent w/ book notation)





$$P\left(t_{n-1}\left(\tfrac{\alpha}{2}\right) \le \frac{\sqrt{n}(\bar{X}-\mu)}{S} \le t_{n-1}\left(1-\tfrac{\alpha}{2}\right)\right) = 1-\alpha$$

...rearrange...

$$P\left(\bar{X} - t_{n-1}\left(1-\tfrac{\alpha}{2}\right)\frac{S}{\sqrt{n}} \le \mu \le \bar{X} - \frac{S}{\sqrt{n}} t_{n-1}\left(\tfrac{\alpha}{2}\right)\right) = 1-\alpha$$

$\therefore$ a 95% C.I. for μ is

$$\left[\bar{X} - t_{n-1}(0.975)\frac{S}{\sqrt{n}},\ \bar{X} - \frac{S}{\sqrt{n}} t_{n-1}(0.025)\right]$$

The MLE for σ^2 is $\hat{\sigma}^{2\,MLE} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

It can be shown that $\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$ (A Chi-squared distribution with $n-1$ degrees of freedom)

Find a $(1-\alpha)*100\%$ CI for σ^2 in terms of the quantiles $\chi^2_{n-1}(\frac{\alpha}{2})$ and $\chi^2_{n-1}(1-\frac{\alpha}{2})$

Hint: Note that if $w \le z$ then $\frac{1}{z} \le \frac{1}{w}$
$(2 \le 3,\ \frac{1}{3} \le \frac{1}{2})$

$$P\left(\chi^2_{n-1}\left(\tfrac{\alpha}{2}\right) \le \frac{n\hat{\sigma}^2}{\sigma^2} \le \chi^2\left(1-\tfrac{\alpha}{2}\right)\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{1}{\chi^2(1-\frac{\alpha}{2})} \le \frac{\sigma^2}{n\hat{\sigma}^2} \le \frac{1}{\chi^2(\frac{\alpha}{2})}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{n\hat{\sigma}^2}{\chi^2(1-\frac{\alpha}{2})} \le \sigma^2 \le \frac{n\hat{\sigma}^2}{\chi^2(\frac{\alpha}{2})}\right) = 1-\alpha$$

$\therefore$ a $(1-\alpha)*100\%$ CI for σ^2 is $\left[\frac{n\hat{\sigma}^2}{\chi^2(1-\frac{\alpha}{2})} \le \sigma^2 \le \frac{n\hat{\sigma}^2}{\chi^2(\frac{\alpha}{2})}\right]$