~~log~~

$$f_{\Theta|X_1,\dots,X_n}(\theta|x_1,\dots,x_n) = \exp\left[\log\left\{f_{\Theta|X_1,\dots,X_n}(\theta|x_1,\dots,x_n)\right\}\right]$$

$$= \exp\left[\log\left\{c\cdot f_{\Theta}(\theta)\cdot\prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta)\right\}\right] \qquad (1)$$

Let's find a 2nd-order Taylor approximation to (at $\hat{\theta}^{MLE}$)

$$\log\left\{\cancel{c}\cdot f_{\Theta}(\theta)\cdot\prod_{i=1}^{n} f_{X_i|\Theta}(x_i|\theta)\right\}$$

$$= \cancel{\log(c)} + \log\{f_{\Theta}(\theta)\} + \sum_{i=1}^{n} \log\{f_{X_i|\Theta}(x_i|\theta)\}$$

By our argument in pictures, $\log\{f_{\Theta}(\theta)\}$ will be nearly constant in comparison to $\sum_{i=1}^{n}\log\{f_{X_i|\Theta}(x_i|\theta)\}$ as a function to θ for large n

$$P_2(\theta) \approx \ell(\hat{\theta}^{MLE}) + \underbrace{\ell'(\hat{\theta}^{MLE})}_{0}(\theta-\hat{\theta}^{MLE}) + \frac{1}{2}\ell''(\hat{\theta}^{MLE})(\theta-\hat{\theta}^{MLE})^2 \qquad (2)$$

Plugging (2) into (1),

$$f_{\Theta|X_1,\dots,X_n}(\theta|x_1,\dots,x_n) \approx \exp\left[\log(c) + \ell(\hat{\theta}^{MLE}) + \frac{-1}{2\left(-\frac{d^2}{d\theta^2}\ell(\theta)\big|_{\theta=\hat{\theta}^{MLE}}\right)}(\theta-\hat{\theta}^{MLE})^2\right]$$

$$\propto \underbrace{\exp\left[\frac{-1}{2\left(-\frac{d^2}{d\theta^2}\ell(\theta)\big|_{\theta=\hat{\theta}^{MLE}}\right)}(\theta-\hat{\theta}^{MLE})^2\right]}_{\text{This is proportional to the pdf of a Normal}\left(\hat{\theta}^{MLE},\ -\left[\frac{d^2}{d\theta^2}\ell(\theta)|_{\theta=\hat{\theta}^{MLE}}\right]^{-1}\right)\text{ distribution}}$$