Newton Raphson for Optimization

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Poisson Model Example

Example A in Section 8.4 of Rice

The National Institute of Science and Technology did a study where they wanted to develop measurement standards for asbestos concentration. Asbestos dissolved in water was spread on a filter, and 3-mm diameter punches were taken from the filter and mounted on a transmission electron microscope. An operator counted the number of fibers in each of 23 grid squares. For the sake of illustration I am using just the first 5 observations here.

Let X_i be the number of fibers of asbestos found in square number i.

Model: $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$

Log-likelihood and its derivatives

$$f(x_i|\lambda) = e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}$$

$$\ell(\lambda|x_1,\ldots,x_n) = \cdots = -n\lambda + \log(\lambda)\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\log(x_i)$$

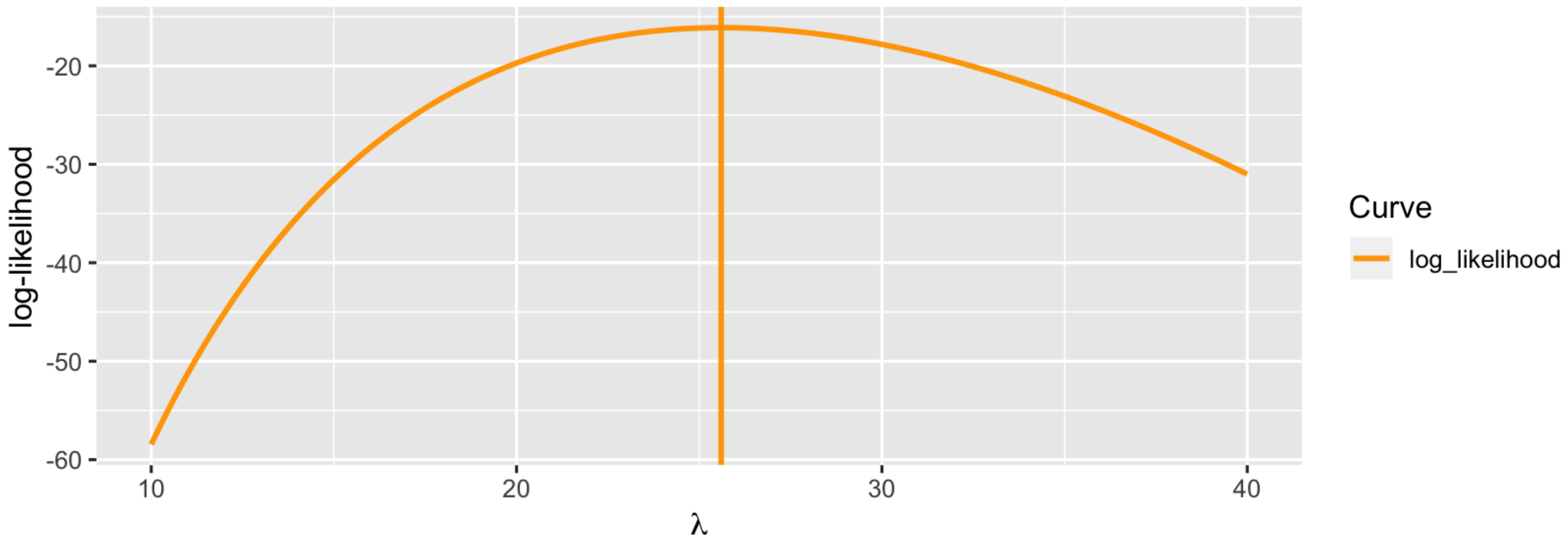
$$\frac{d}{d\lambda}\ell(\lambda|x_1,\ldots,x_n) = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n$$

$$\frac{d^2}{d\lambda^2}\ell(\lambda|x_1,\ldots,x_n) = \frac{-1}{\lambda^2}\sum_{i=1}^{n} x_i$$

We saw previously that $\hat{\lambda}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$

In our example, $\hat{\lambda}_{MLE} = 25.6$

Log-likelihood function, MLE



The vertical orange line is at the MLE.

…But what if we couldn't solve for the MLE directly?

Taylor Series Approximation to L

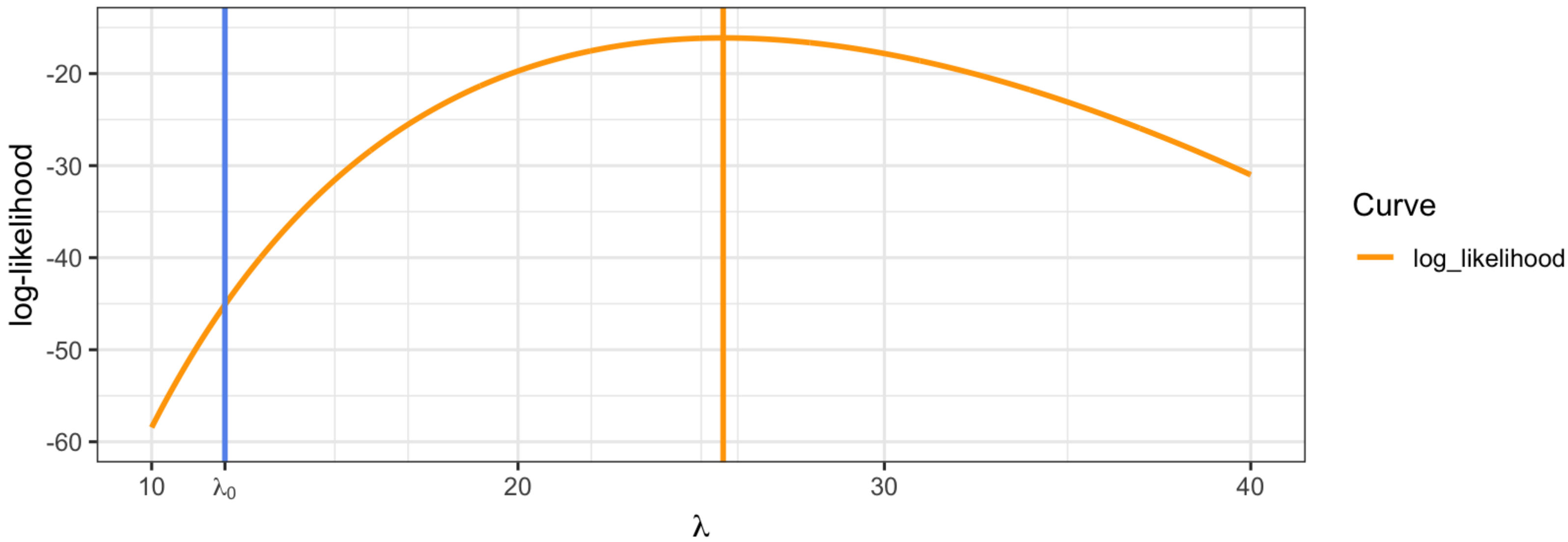
Pick a value λ_0. The second-order Taylor Series approximation to $L(\lambda|x_1,\ldots,x_n)$ around λ_0 is

$$\begin{aligned} P_2(\lambda) &= L(\lambda_0|x_1,\ldots,x_n) + \frac{d}{d\lambda}L(\lambda_0|x_1,\ldots,x_n)(\lambda-\lambda_0) \\ &+ \frac{1}{2}\frac{d^2}{d\lambda^2}L(\lambda_0|x_1,\ldots,x_n)(\lambda-\lambda_0)^2 \end{aligned}$$

The maximum of $P_2(\lambda)$ is at $\lambda_1 = \lambda_0 - \frac{\frac{d}{d\lambda}L(\lambda_0|x_1,\ldots,x_n)}{\frac{d^2}{d\lambda^2}L(\lambda_0|x_1,\ldots,x_n)}$

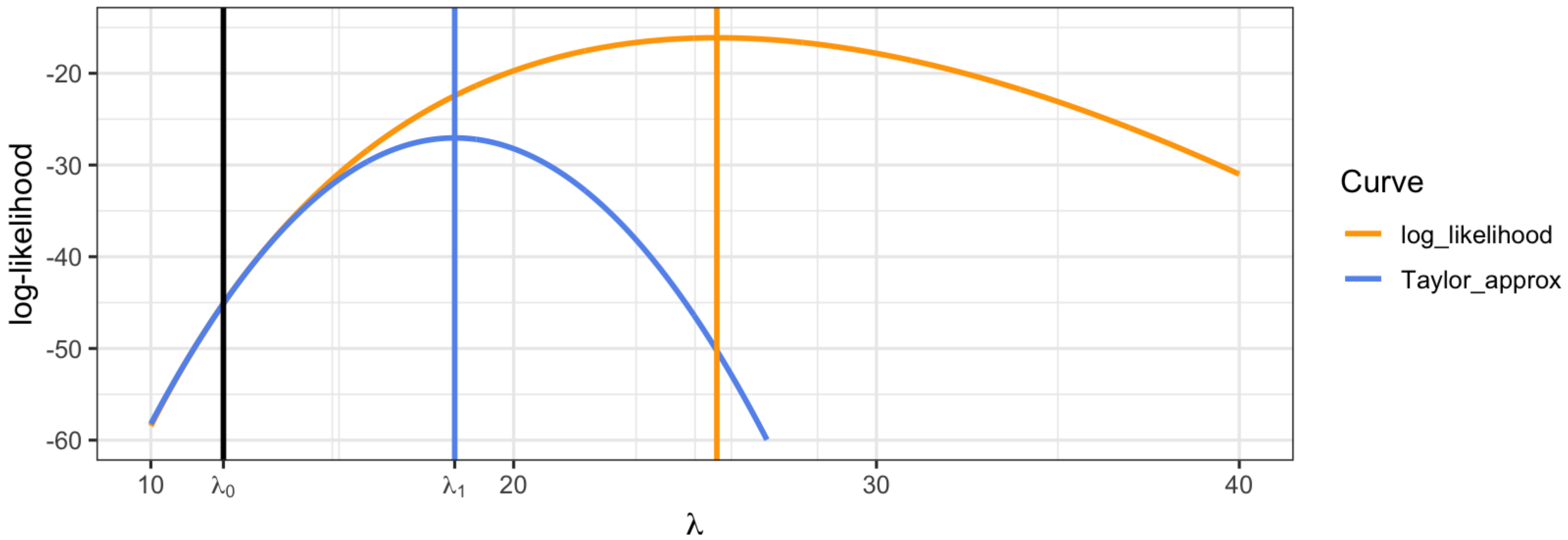
Now repeat, but centering the Taylor Series approximation at λ_1.

Pick λ_0



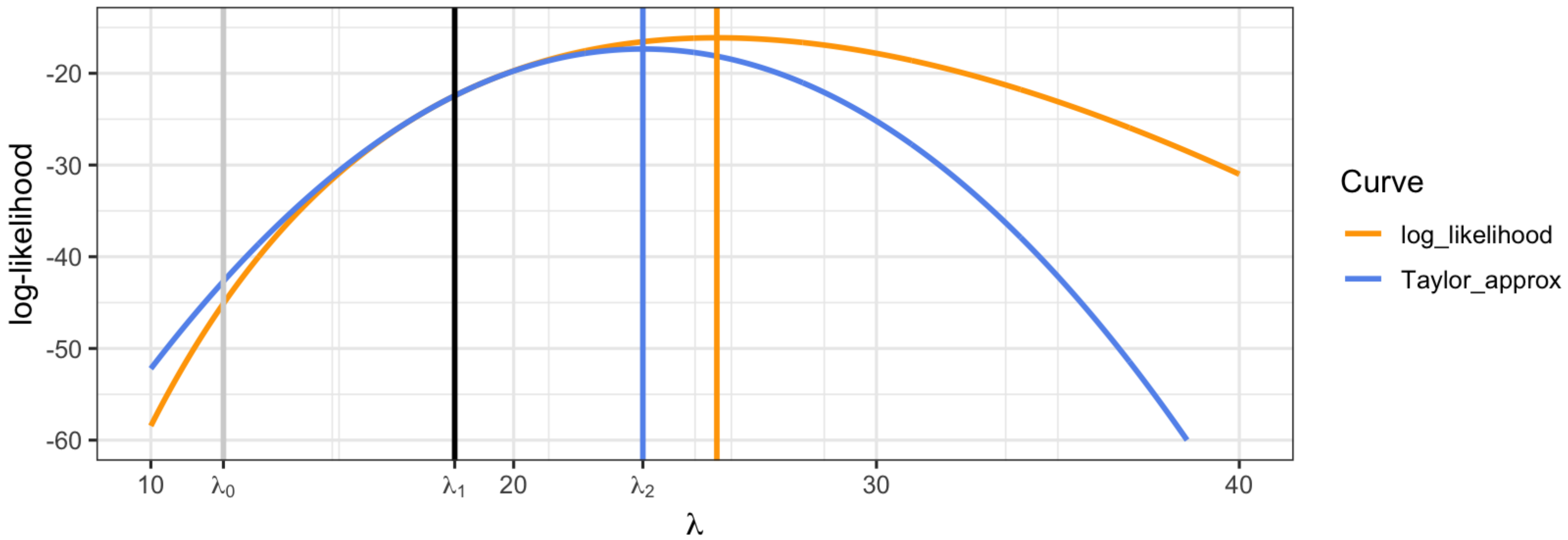
$$\lambda_0 = 12$$

Approximate L around λ_0, get λ_1



$$\lambda_0 = 12, \lambda_1 = \lambda_0 - \frac{\frac{d}{d\lambda} L(\lambda_0 | x_1, \ldots, x_n)}{\frac{d^2}{d\lambda^2} L(\lambda_0 | x_1, \ldots, x_n)} = 18.375$$

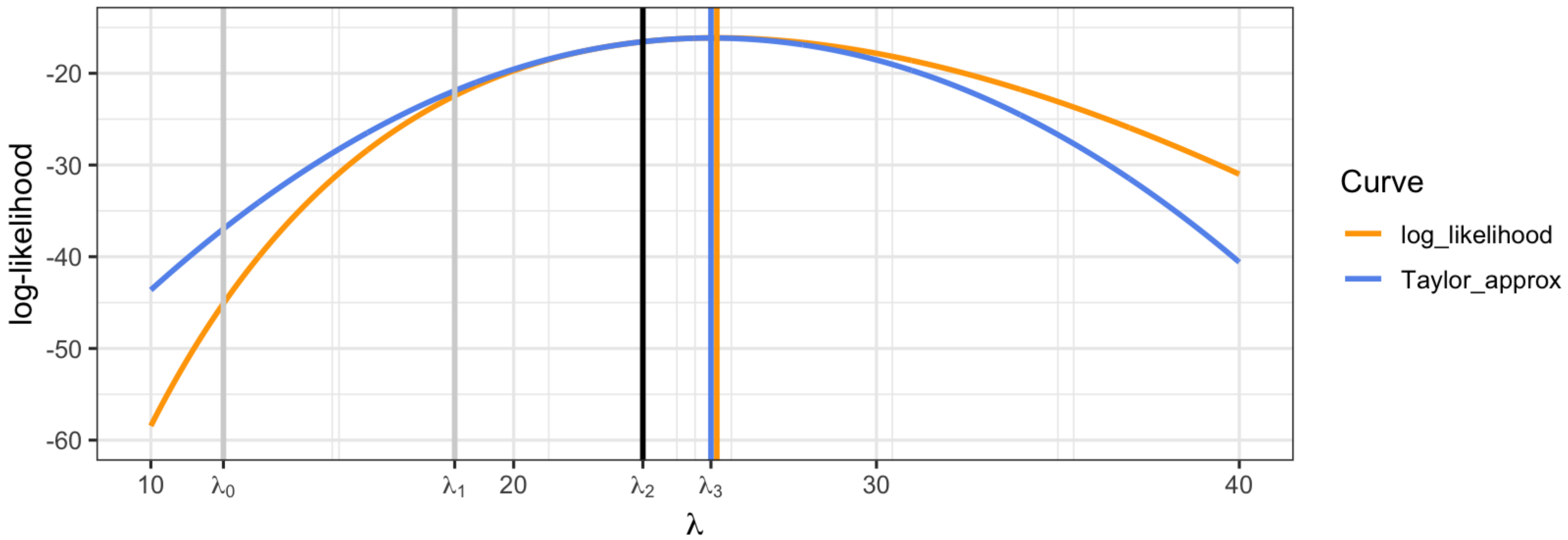
Approximate L around λ_1, get λ_2



$$\lambda_0 = 12,\ \lambda_1 = 18.375,$$

$$\lambda_2 = \lambda_1 - \frac{\frac{d}{d\lambda} L(\lambda_1 | x_1, \ldots, x_n)}{\frac{d^2}{d\lambda^2} L(\lambda_1 | x_1, \ldots, x_n)} = 23.561$$

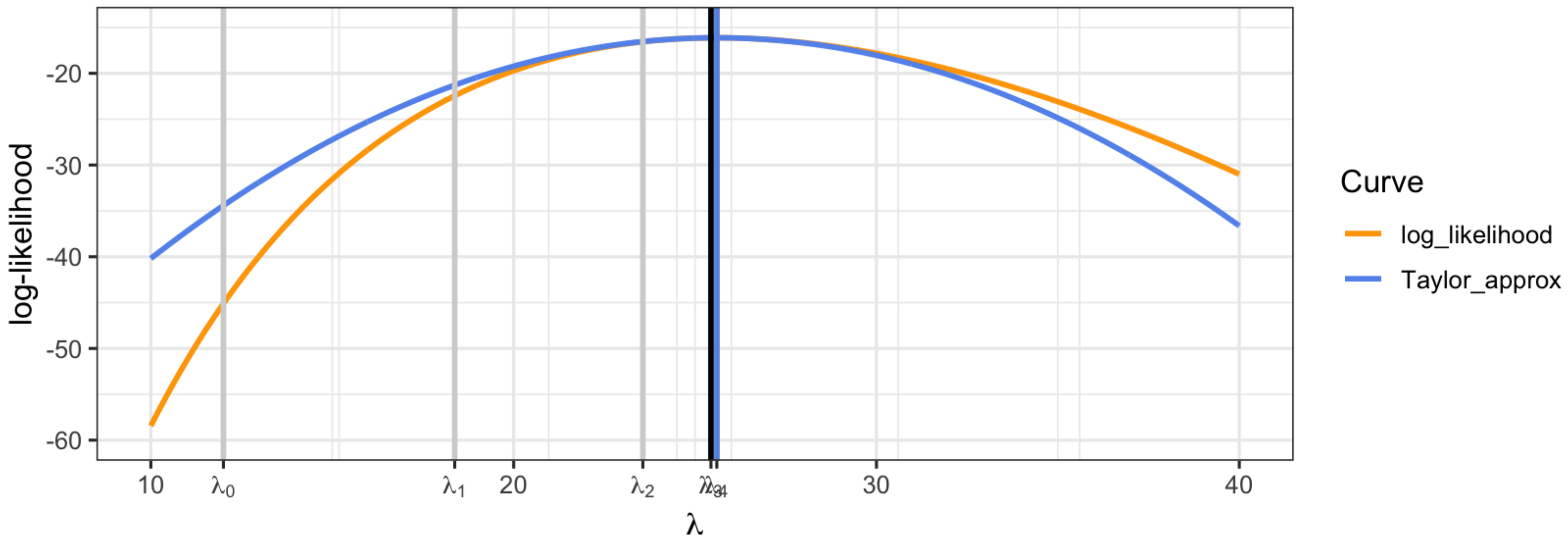
Approximate L around λ_2, get λ_3



$$\lambda_0 = 12,\ \lambda_1 = 18.375, \lambda_2 = 23.561$$

$$\lambda_3 = \lambda_2 - \frac{\frac{d}{d\lambda} L(\lambda_2 | x_1, \ldots, x_n)}{\frac{d^2}{d\lambda^2} L(\lambda_2 | x_1, \ldots, x_n)} = 25.438$$

Approximate L around λ_3, get λ_4



$$\lambda_0 = 12,\ \lambda_1 = 18.375, \lambda_2 = 23.561, \lambda_3 = 25.438$$

$$\lambda_4 = \lambda_3 - \frac{\frac{d}{d\lambda} L(\lambda_3 | x_1, \dots, x_n)}{\frac{d^2}{d\lambda^2} L(\lambda_3 | x_1, \dots, x_n)} = 25.599$$