Results so far:

$E[\bar{X}] = \mu$

$Var(\bar{X}) = \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right)$

If n and N are "large enough", it's approximately true that $\bar{X} \sim \text{Normal}\left(\mu, \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right)\right)$

$s^2\left(1 - \frac{1}{N}\right) = \frac{1}{n-1}\left(1 - \frac{1}{N}\right)\sum_{i=1}^{n}(x_i - \bar{x})^2$

is an unbiased estimator of σ^2

Define

$$\hat{\sigma}^2_{\bar{X}} = \frac{s^2}{n}\left(1 - \frac{n}{N}\right)$$

$$E\left[\hat{\sigma}^2_{\bar{X}}\right] = E\left[\frac{s^2}{n}\left(1 - \frac{1}{N}\right) \cdot \frac{N}{N-1}\left(\frac{N-n}{N}\right)\right]$$

$$= \frac{1}{n} \cdot \frac{N-n}{N-1} \cdot \sigma^2$$

$$= \frac{\sigma^2}{n}\left(1 - \frac{n-1}{N-1}\right)$$

Unbiased estimate of $Var(\bar{X})$

If n and N are large enough, it's approximately true that

$$\bar{X} \sim \text{Normal}\left(\mu, \frac{s^2}{n}\left(1 - \frac{n-1}{N-1}\right)\right)$$

$$\bar{X} \sim \text{Normal}\left(\mu, \hat{\sigma}^2_{\bar{X}}\right)$$

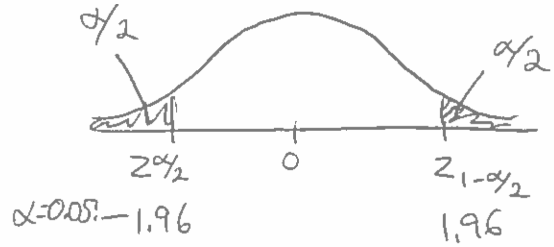
Suppose the approximation is good.

$$\frac{\bar{X} - \mu}{\hat{\sigma}_{\bar{X}}} \sim \text{Normal}(0, 1)$$

Our goal: A ~~random~~ pair of random variables A, B s.t. $P(A \le \mu \le B) = 1-\alpha$

For a particular sample, the realized values a, b will give a confidence interval.

$$P\left(z_{\alpha/2} \le \frac{\bar{X} - \mu}{\hat{\sigma}_{\bar{X}}} \le z_{1-\alpha/2} \right) = \alpha$$



$$\Rightarrow \quad P\left(z_{\alpha/2}\hat{\sigma}_{\bar{X}} \le \bar{X} - \mu \le z_{1-\alpha/2}\hat{\sigma}_{\bar{X}} \right) = \alpha$$

$$\Rightarrow \quad P\left(-\bar{X} + z_{\alpha/2}\hat{\sigma}_{\bar{X}} \le -\mu \le -\bar{X} + z_{1-\alpha/2}\hat{\sigma}_{\bar{X}} \right) = \alpha$$

$$\Rightarrow \quad P\left(\bar{X} - z_{\alpha/2}\hat{\sigma}_{\bar{X}} \ge \mu \ge \bar{X} - z_{1-\alpha/2}\hat{\sigma}_{\bar{X}} \right) = \alpha$$

$$\Rightarrow \quad P\left(\bar{X} - z_{1-\alpha/2}\hat{\sigma}_{\bar{X}} \le \mu \le \bar{X} - z_{\alpha/2}\hat{\sigma}_{\bar{X}} \right) = \alpha$$

random variable

Set $A = \bar{X} - z_{1-\alpha/2}\hat{\sigma}_{\bar{X}}$, $B = \bar{X} - z_{\alpha/2}\hat{\sigma}_{\bar{X}}$

The <u>random interval</u> $[A, B]$ is an approximate $(1-\alpha)100\%$ CI for μ.

realized value

For a given sample $[\bar{x} - z_{1-\alpha/2}\hat{\sigma}_{\bar{x}}, \ \bar{x} - z_{\alpha/2}\hat{\sigma}_{\bar{x}}]$ is the <u>realized value</u> of this interval.