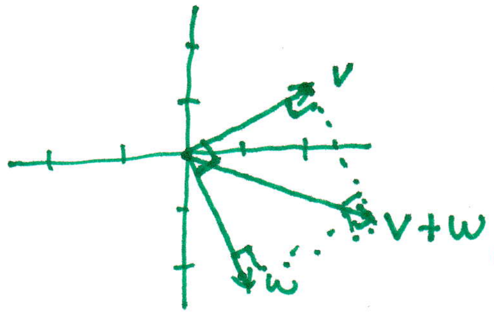
Reminder about orthogonal vectors and projections:

Def.: Vectors v and w are orthogonal (perpendicular) if their inner product is 0. Denote by $v \perp w$

Ex.: $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$$v \cdot w = v'w = [2 \ \ 1] \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 2 \cdot 1 + 1 \cdot (-2) = 0$$

Why?

- If v and w are perpendicular, the triangle is a right triangle
- Pythagorean thm says $\|v\|^2 + \|w\|^2 = \|v+w\|^2$

$$\Rightarrow \left(\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}\right)^2 + \left(\sqrt{w_1^2 + w_2^2 + \cdots + w_n^2}\right)^2 = \left(\sqrt{(v_1+w_1)^2 + \cdots + (v_n+w_n)^2}\right)^2$$

$$\Rightarrow v_1^2 + v_2^2 + \cdots + v_n^2 + w_1^2 + \cdots + w_n^2 = v_1^2 + 2v_1w_1 + w_1^2 + \cdots + v_n^2 + 2v_nw_n + w_n^2$$

$$\Rightarrow 0 = 2v_1w_1 + \cdots + 2v_nw_n$$

$$\Rightarrow 0 = 2v'w$$

$$\Rightarrow v'w = 0$$

Def.: The column space of a matrix $X = \begin{bmatrix} | & | & & | \\ X_{\cdot,1} & X_{\cdot,2} & \cdots & X_{\cdot,p} \\ | & | & & | \end{bmatrix}$

is the linear span of its columns:

$$\mathcal{C}(X) = \left\{ \underset{\sim}{v} : v = \sum_{j=1}^{p} a_j X_{\cdot,j} \right\}$$

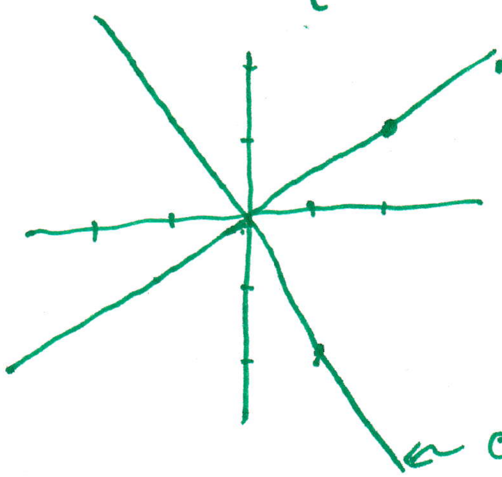
Def.: H is a <u>perpendicular projection operator</u> (matrix) onto $\mathcal{C}(X)$ if and only if

(i) $\underset{\sim}{v} \in \mathcal{C}(X) \Rightarrow H\underset{\sim}{v} = \underset{\sim}{v}$ (projection - doesn't change things in $\mathcal{C}(X)$)

(ii) $\underset{\sim}{w} \perp \mathcal{C}(X) \Rightarrow H\underset{\sim}{w} = \underset{\sim}{0}$ (perpendicularity - vectors orthogonal to $\mathcal{C}(X)$ go to 0)

Ex.: Suppose $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\mathcal{C}(X) = \left\{ \underset{\sim}{v} : \underset{\sim}{v} = a \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ → things like $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$



← line is $\mathcal{C}(X)$

← orthogonal complement to $\mathcal{C}(X)$ is vectors of the form $a\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Let $H = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$, which is the perpendicular projection operator onto $\mathcal{C}(X)$.

Verify: (i) For any $a \in \mathbb{R}$,

$$H \begin{bmatrix} 2a \\ a \end{bmatrix} = \begin{bmatrix} (0.8 \cdot 2a + 0.4a) \\ 0.4 \cdot 2a + 0.2a \end{bmatrix} = \begin{bmatrix} 2a \\ a \end{bmatrix}$$ (property i holds)

(i) For any $a \in \mathbb{R}$,

$$H \begin{bmatrix} a \\ -2a \end{bmatrix} = \begin{bmatrix} 0.8 \cdot a + 0.4(-2a) \\ 0.4a + 0.2(-2a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thm:

The (unique) perpendicular projection operator onto $\mathcal{C}(X)$ is

$$X(X'X)^{-1}X'.$$

Proof: Long and not helpful for intuition.
Basically, verify conditions (i) and (ii) in the definition.

Ex: If $X = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$X(X'X)^{-1}X' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\left([2\ 1]\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)^{-1}[2\ 1] = \cdots = \begin{bmatrix} 0.8 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$$